A note on regularity and rigidity of co-dimension 1 Sobolev isometric immersions

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Abstract

We prove the C^1 regularity and developability of $W^{2,m}$ Sobolev isometric immersions of m-dimensional domains into $\mathbb{R}^{m+1}$. A corollary is the strong density of smooth mappings in this class when the domain is convex. We also prove that any $W^{2,n}$-isometric immersion of $\mathbb{S}^n$ inside $\mathbb{S}^{n+1}$ is a rigid motion.

1 Introduction

Function spaces with constraint arise as an important tool in the study of qualitative features of solutions to various nonlinear and geometric PDEs such as singularities and oscillations and the opposite phenomena such as regularity, rigidity, compactness and convergence. An important feature in the study of the mappings in these spaces is the interaction between their analytical and geometric or topological properties. In this paper we consider certain classes of Sobolev mappings with isometry constraint. These constraints are naturally of geometric nature, on the derivatives, and of non-convex nature. The main goal of our investigation is to determine whether the rigidity properties of smooth isometric immersions [11] are inherited by isometries of a weaker class of regularity.

Rigidity results in classical differential geometry depend heavily on the regularity of the given mapping. For example, Kuiper showed that there are C^1 smooth isometric embeddings of the unit sphere S^2 into arbitrary small balls in $\mathbb{R}^3$ [7], while Hilbert had already shown that a C^2 smooth isometric immersion of the sphere in $\mathbb{R}^3$ is a rigid motion. Hence, from an analytical point of view, it is natural to study the isometric immersions which enjoy a somewhat intermediate regularity, e.g. of Sobolev type.

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Density of a suitable class of "good" functions in the space defined by a functional is a valuable tool in calculus of variations. Usually, a smoother class of functions constitute the natural class of good functions to consider. The density results can be used e.g. in proving regularity results for the critical points or in controlling the energy of the recovery sequences in the context of Γ-convergence, for example when convergence is studied in the reduction from thin three-dimensional nonlinear elasticity to two-dimensional plate or shell theories. In several instances, this question is naturally connected to the topological and geometric rigidity properties of the smoother functions. A major indication of a positive answer to the density question is when the classical rigidity results are true for mappings of Sobolev type.

In [9] the author proved that if $\Omega \subset \mathbb{R}^2$ is a Lipschitz domain, any isometric immersion in $W^{2,2}(\Omega, \mathbb{R}^3)$ is of class C^1 and is developable, i.e. it satisfies the following property. For any $x \in \Omega$, either u is affine in a neighborhood of x in Ω, or it is affine on a segment passing through x and joining Ω on both sides. Also, it was proved that if the domain Ω is convex, smooth isometries are dense in the space of $W^{2,2}$ isometries. This result was generalized by Hornung [4, 5] to other classes of domains and by Jerrard to the class of Monge-Ampère functions [6]. Note that the Sobolev exponent $p = 2$ is the borderline regularity for which these results hold true. In particular, the mapping $u(r\cos\theta, r\sin\theta) = r(1/2\cos 2\theta, 1/2\sin 2\theta, \sqrt{3}/2)$ is an isometric immersion of the unit disk in $\mathbb{R}^2$ with a conic singularity at the origin and enjoying $W^{2,p}$ regularity for all $p < 2$, which cannot be approximated by smooth isometries. Note that this mapping is a one-homogeneous extension of a smooth isometric immersion of $\mathbb{S}^1$, the unit circle, into $\mathbb{S}^2$, the unit sphere. Compare with the situation in higher dimensions explained in Remark 1.1.

Here, we will generalize the results of [9] to a higher dimensional setting. We will focus on the isometric immersions of an m dimensional domain into $\mathbb{R}^{m+1}$ which enjoy a borderline $W^{2,m}$ regularity. We will discuss questions of regularity, rigidity and approximability by smooth mappings in these classes. Similar problems for $W^{2,m}$-regular Sobolev isometric immersions of m-dimensional domains into Euclidean spaces of dimension bigger than $m + 1$ are still open.

Note that, by Sobolev embedding theorems, $W^{2,m}$ mappings of m dimensional domains are just short of being in C^1. Our first result, is the following extra regularity gain for isometric mappings.

Theorem 1. *Let $\Omega \subset \mathbb{R}^m$ be a bounded Lipschitz domain and let $u : \Omega \to \mathbb{R}^{m+1}$ be an isometric immersion of class $W^{2,m}$, i.e. $u \in I^{2,m}(\Omega, \mathbb{R}^{m+1})$.*

Then u is of C^1 regularity inside Ω.

This regularity result is a crucial ingredient in our analysis. Combined with the developability results of [9], it yields the main result of this paper:

Theorem 2. *Let Ω and u be as above. For any $x \in \Omega$, either u is affine in a neighborhood of x, or there exists a unique hyperplane $P \ni x$ of $\mathbb{R}^n$ such that u is affine on the connected component of x in $\Omega \cap P$.*

In the course of the proof we will study of infinitesimal isometries on a flat domain. The main observation is that $u : \Omega \to \mathbb{R}^{m+1}$ is an isometric immersion only if all its components are second order infinitesimal isometries. Consider a vector field $v : \Omega \to \mathbb{R}^{m+1}$ which satisfies the following property: There exists a vector field $w : \Omega \to \mathbb{R}^{m+1}$ for which the deformation $\phi_\varepsilon := \mathrm{id} + \varepsilon v + \varepsilon^2 w$ induces a change of metric of at most order ε^3 on Ω. Through a straightforward calculation, it can be shown that V is a second order infinitesimal isometry if and only if $(v_1, \cdots, v_m)$ is an affine map with skew symmetric gradient and all the 2-minors of the Hessian matrix $\nabla^2 v_{m+1}$ vanish in Ω. We will prove the equivalent results of regularity and developability for $W^{2,m}$ second order infinitesimal isometries in Propositions 7 and 8. These propositions will be the main ingredients of the proofs when we will later study the isometric immersions of spherical domains into spherical targets.

The developability property proved in 2 is the key to prove the density of smooth isometries in the same class of isometries. Indeed, our conjecture is that the developability properties proved in [11] for smooth isometric immersions of an m-dimensional domain into $\mathbb{R}^k$ for $k < 2m$ are the sufficient conditions for approximability of Sobolev isometries by smooth isometries. We will tackle this more general problem in future. However the following statement can be proved in a straightforward manner following the same footsteps for the case $m = 2$ in [9]. We will leave the details to the patient reader.

Theorem 3. *Let Ω be a convex domain of $\mathbb{R}^m$. Then smooth isometric immersions are strongly dense in the space of $W^{2,m}$ isometric immersions from Ω into $\mathbb{R}^{m+1}$.*

In what follows we will study isometric immersions of spherical domains into higher dimensional spheres. As a preliminary, we first introduce some definitions.

Given an m-dimensional compact smooth Riemannian manifold (M, g) and $f : M \to R$, we say $f \in W^{k,p}(M)$ if for any coordinate chart (ξ, U)

of M and for any $U' \Subset U$ we have $f \circ \xi^{-1} \in W^{k,p}(\xi(U'))$. Note that this definition does not depend on the choice of g. Since M is compact we can find a finite cover $\left\{U'_i \Subset U_i\right\}_{i=1}^n$ and maps $\xi_i : U_i \to \mathbb{R}^m$ such that $(\xi_i, U_i)_{i=1}^n$ is an atlas for M. Let the Banach norm of $W^{k,p}(M)$ be

$$\|u\|_{k,p,M} := \sum_{i=1}^{n} \|f \circ \xi_i^{-1} \restriction_{\xi_i(U'_i)} \|_{k,p}. \tag{1}$$

The topology of the Banach space $W^{k,p}(M)$ does not depend on the choice of the finite cover. We define then $W^{k,p}(M, \mathbb{R}^k)$ to be the space of all $\mathbb{R}^k$ valued mappings from M whose components are in $W^{k,p}(M)$.

Let (N, h) be a smooth Riemannian manifold. We say the Lipschitz mapping $u : M \to N$ is an isometric immersion and we denote it by $I^{1,\infty}(M, N)$ if for almost every $x \in M$, the tangent map $du : TM \to TN$ satisfies $\langle X|Y\rangle_g = \langle du(x)X|du(x)Y\rangle_h$, for all $X, Y \in T_xM$.

Finally, let (N, h) be isometrically embedded in some Euclidean space $\mathbb{R}^k$. We introduce the space of $W^{k,p}$-isometric immersions between M and N to be

$$I^{k,p}(M, N) := I^{1,\infty}(M, N) \cap W^{k,p}(M, \mathbb{R}^k).$$

This function space is independent of the specific isometric embedding of N we chose for the definition.

It is sometimes preferable to work with an equivalent norm. Indeed the norm introduced in (1) is not an intrinsic norm, as the natural bending of N inside $\mathbb{R}^k$ participates in the L^2 norm of the second derivatives of u. We write ∇ to denote the covariant derivative with respect to the metric g on M. We would like to introduce the space $I^{2,p}_{in}(M, N)$ of all isometric immersions $u : M \to N$ with finite p-bending energy:

$$E_{b,p}(u) = \|D^2u\|^p_{p,in} := \int_M \left| P_{T_{u(x)}N} \left(\nabla^2 u(x)\right)\right|^p dvol_M.$$

Here T_yN is the tangent space to N at the point y and P_E is the orthogonal projection over the linear subspace E of $\mathbb{R}^k$. Note that ∇ is a $C^\infty(M)$-linear operator over the smooth vector fields of M with value in $\mathbb{R}^k$, and the value of $X \cdot \nabla^2 u(x) \in \mathbb{R}^k$ depends only on the value of the vector field X at $x \in M$ and the value of u in a neighborhood of x. Hence for almost all x, $\nabla^2 u(x)$ is a linear map from T_xM to $\mathbb{R}^k$ and its projection over a subspace of $\mathbb{R}^k$ is well defined almost everywhere. We use the usual linear operator norm (with respect to the scalar product induced by g on T_xM). The intrinsic norm for $u \in I^{2,p}(M,N)$ is

$$\|u\|_{2,p,in} := \|u\|_{W^{1,p}(M)} + \|D^2u\|_{p,in}. \tag{2}$$

We denote by $\mathbb{S}^n$ the standard n-dimensional sphere and by $i : \mathbb{S}^n \to \mathbb{S}^{n+1}$ the natural embedding of $\mathbb{S}^n \subset \mathbb{R}^n$ inside $\mathbb{S}^{n+1} \subset \mathbb{R}^{n+2}$ defined by $i(x_1, \cdots, x_{n+1}) = (x_1, \cdots, x_{n+1}, 0)$. Let U be an open subset of $\mathbb{S}^n$. We say $r : U \to \mathbb{S}^{n+1}$ is a rigid motion if $r = Ri$ for some $R \in SO(n+2)$. We say $S \subset \mathbb{S}^{n+1}$ is a hypersphere if it is the image of a rigid motion $r : \mathbb{S}^n \to \mathbb{S}^{n+1}$. Note that for any rigid motion $r : \mathbb{S}^n \to \mathbb{S}^{n+1}$, $E_b(r) = 0$. The following theorem generalized Theorems 1 and 2 to isometric immersions of spherical domains.

Theorem 4. *Let S be a domain in $\mathbb{S}^{n+1}$ and $u \in I^{2,n}(S, \mathbb{S}^{n+1})$. Then $u \in C^1(S)$. Moreover, to any $x \in U$, we can associate a rigid motion $r(x) : \mathbb{S}^n \to \mathbb{S}^{n+1}$ with the following property. Let*

$$S_0 := \{x \in S : \ u = r(x) \ \text{on a neighborhood of } x\}.$$

Then for each $x \in S \setminus S_0$, there exists a unique hypersphere $P(x)$ such that u coincides with $r(x)$ on the connected component of x in $S \cap P(x)$.

A similar result to Theorem 3 follows:

Corollary 5. *Let U be a geodesically convex domain of $\mathbb{S}^n$. Then smooth isometric immersions are strongly dense in $I^{2,n}(S, \mathbb{S}^{n+1})$.*

Finally, we obtain the following strong rigidity result for $n > 1$.

Corollary 6. *Any $W^{2,n}$ isometric immersion of $\mathbb{S}^n$ inside $\mathbb{S}^{n+1}$ is a rigid motion.*

Remark 1.1. *In view of this corollary, we cannot construct an isometric immersion in $W^{2,2}(B^3, \mathbb{R}^4)$ with a conic singularity at the origin as we did for the subcritical Sobolev exponent in [9]. The question of developability or approximability by smooth mappings in $I^{2,2}(B^3, \mathbb{R}^4)$ remains open.*

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2 C^1 regularity

In this section we intend to prove Theorem 1.

Proposition 7. *Let $V \in W^{2,m}(\Omega)$ such that rank$(\nabla^2 V) \leq 1$ a.e. in Ω, then $V \in C^1(\Omega)$.*

Proof: Let $f = \nabla V$ and Since rank $(\nabla^2 V) \leq 1$, all the 2×2 minors of ∇f vanish almost everywhere in Ω. Hence for $1 \leq i < j \leq m$, $f^i_{,i} f^j_{,j} - f^i_{,j} f^j_{,i} = 0$ a.e. in Ω, where the subscript $,i$ denotes the partial derivative in the ith direction. Moreover by the symmetry of the $\nabla^2 V$ we have $f^i_{,j} = f^j_{,i}$. For $\delta > 0$, let the mapping $f_\delta : \Omega \to \mathbb{R}^m$ be defined by $f_\delta := f + \delta(-x_j e_i + x_i e_j)$. We have $f_\delta \in W^{1,m}(\Omega, \mathbb{R}^m)$ and a simple calculation shows that

$$M_2^{ij}(f) = (f^i_\delta)_{,i}(f^j_\delta)_{,j} - (f^i_\delta)_{,j}(f^j_\delta)_{,i} = \delta^2 > 0.$$

In order to prove the proposition, it is sufficient to who that $h = f^i_\delta$ is continuous. We prove first the weak monotonicity of h. The argument is similar to the proof of weak monotonicity of deformations with positive Jacobian due to J. Manfredi [8] (see also [1], page 119). Let $B \subset\subset \Omega$ be an open ball and let $M = \sup_{\partial B} f^i$ and $g = \max\{h, M\}$. Note that $g = M$ on ∂B. We have

$$I = \int_B (g_{,i}(f^j_\delta)_{,j} - g_{,j}(f^j_\delta)_{,i} = 0$$

since $g = M$ on ∂B and the integrand is Null-Lagrangian. On the other hand

$$I = \int_{\{x\in B: h(x) > M\}} (h)_{,i}(f^j_\delta)_{,j} - (h)_{,j}(f^j_\delta)_{,i} = \delta^2 |\{x \in B : h(x) > M\}|.$$

As a consequence $|\{x \in B : h(x) > M\}| = 0$. Similarly we obtain that

$$|\{x \in B : h(x) < \inf_{\partial B} h\}| = 0.$$

Hence, for almost all $(x, y) \in B \times B$, $|h(x) - h(y)| \leq \operatorname{diam} h(\partial B)$. In other words

$$\operatorname{ess\,osc}_B h := \operatorname{ess\,sup}_{B\times B} |h(x) - h(y)| \leq \operatorname{diam} h(\partial B). \tag{3}$$

Now let $0 < r < R$ and let $B_R = B_R(x) \subset \Omega$. By the Sobolev embedding $W^{1,m}(\partial B_1) \to C^{0,1/m}(\partial B_1)$, we have after rescaling

$$\sup_{\partial B_\rho} |u(x) - u(y)| \leq C\rho \Big(\int_{\partial B_\rho(x)} |\nabla u|^m \, d\mathcal{H}^{m-1} \Big)^{1/m}.$$

Therefore

$$
\begin{aligned}
\ln(R/r)(\operatorname{ess\,osc}_{B_r} h)^m &\le \int_r^R \frac{1}{\rho} |\operatorname{ess\,osc}_{B_\rho} h|^m d\rho \\
&\le \int_r^R \frac{1}{\rho} |\operatorname{diam} h(\partial B_\rho(x))|^m d\rho \\
&\le C \int_r^R \Big(\int_{\partial B_\rho(x)} |\nabla h|^m \, d\mathcal{H}^{m-1} \Big) \, d\rho \\
&\le C \int_{B_R \setminus B_r} |\nabla h|^m .
\end{aligned} \tag{4}
$$

Hence we obtain

$$
\operatorname{ess\,osc}_{B_r(x)} h \le C_R |\ln r|^{-1/m} \to 0 \quad \text{as } r \to 0, \tag{5}
$$

since $h \in W^{1,m}(\Omega)$. Applying (5) to $y \in B_{R/2}(x) \subset \Omega$ and $0 < r, r' < R/2$ we have

$$
\begin{aligned}
\left| \fint_{B_r(y)} h - \fint_{B_{r'}(y)} h \right| &\le \fint_{B_r(y) \times B_{r'}(y)} |h(x) - h(y)| \, dx dy \\
&\le C_R |\ln|^{-1/m} (\max\{r, r'\}) \to 0 \quad \text{if } r, r' \to 0.
\end{aligned} \tag{6}
$$

Hence the proper representation of h

$$
h^*(y) := \lim_{r \to 0} \fint_{B_r(y)} h,
$$

is well defined everywhere in $B_{R/2}(y)$. Replacing h by h^* and passing to the limit $r' \to 0$ in (6) we obtain

$$
|\fint_{B_r(y)} h - h(y)| \le C_R |\ln r|^{-1/m} \to 0 \quad \text{as } r \to 0.
$$

As a consequence, h is locally a uniform limit of continuous functions and hence continuous. □

Let $u : \Omega \to \mathbb{R}^k$ be a Lipschitz immersion of Ω into $\mathbb{R}^k$, and let $g_{ij} = u_{,i} \cdot u_{,j}$ be the metric associated with u, where the subscript $,i$ denotes the partial derivative $\frac{\partial}{\partial x_i}$.

Lemma 2.1. *Let $u : \Omega \to \mathbb{R}^k$ be a smooth immersion. Then for arbitrary $1 \le i, j, k, l \le m$*

$$
g_{ij,kl} + g_{kl,ij} - g_{il,kj} - g_{kj,il} = -2u_{,ij} \cdot u_{,kl} + 2u_{,il} \cdot u_{,kj}, \tag{7}
$$

and

$$(u_{ij} \cdot \mathbf{n})_{,k} - (u_{ik} \cdot \mathbf{n})_{,j} = u_{,ij} \cdot \mathbf{n}_{,k} - u_{,ik} \cdot \mathbf{n}_{,j} \tag{8}$$

for any vector field $\mathbf{n} \in W^{1,2}(\Omega, \mathbb{R}^k)$.

Proof: By twice differentiating g_{ij} we have the following identity:

$$g_{ij,kl} = u_{,ikl} \cdot u_{,j} + u_{ik} \cdot u_{,jl} + u_{,il} \cdot u_{,jk} + u_{,i} \cdot u_{,jkl}.$$

Making the summation over the proper permutations of i, j, k and l proves identity (7). The second identity is straightforward. $\square$

Let $u \in I^{2,m}(\Omega, \mathbb{R}^{m+1})$. For almost every $\mathbf{x} \in \Omega$ consider the unique unit vector $\mathbf{n}(\mathbf{x}) \in \mathbb{R}^{m+1}$ which is orthogonal to all $u_{,i}$ and satisfies

$$\det[u_{,1} \cdots u_{,m}\, \mathbf{n}] > 0.$$

Note that if $\mathbf{n} = \sum n^i \mathbf{e}_i$ we have

$$\sum_{i=1}^{m+1} n^i \omega_i = \bigwedge_{i=1}^{m+1} \star u_{,i}$$

where

$$\omega_i = (-1)^i \bigwedge_{j \neq i} dx^j$$

and the 1-form $\star u_{,i} = \sum \alpha^i_j dx^j$ is given by $\alpha^i_j = u^j_{,i}$. As a consequence $\mathbf{n} \in W^{1,m}(\Omega, \mathbb{R}^{m+1})$. It is straightforward to observe that there is $A \in L^m(\Omega, M^{m\times m})$ such that $\nabla^2 u = -A\mathbf{n}$. Indeed $A = (\nabla u)^T \nabla \mathbf{n}$ is the so called second fundamental form of u.

Lemma 2.2. *For any simply-connected subdomain* $U \subset \Omega$ *there exists* $f \in W^{1,m}(U, \mathbb{R}^m)$ *for which* $A = \nabla f$. *On the other hand all the 2-minors of* A *vanish.*

Remark 2.1. *As* A *is symmetric, this implies the local existence of* $V \in W^{2,m}$ *whose Hessian is* A.

Proof: Consider a sequence of smooth mappings $u_n : \Omega \to \mathbb{R}^{m+1}$ such that $u_n \to u$ strongly in $W^{2,m}(\Omega, \mathbb{R}^{m+1})$. Apply Lemma 2.1 and pass to the limit. (8) is established for u in the sense of distributions. However, since $\mathbf{n}$ is a unit vector field, $\mathbf{n}_{,i} \in L^2$ is orthogonal to $\mathbf{n}$, hence to $u_{,ij} \in L^2$. This yields

$$A_{ij,k} = A_{ik,j}$$

which, by Poincaré's lemma, establishes the first claim. In the same manner (7) is established in the sense of distributions for u. However since u is an isometric immersion $g_{ij} = \delta_{ij}$ is constant in Ω. Hence for all $1 \leq i,j,k,l \leq m$

$$u_{,ij} \cdot u_{,kl} - u_{,il} \cdot u_{,kj} = 0.$$

Use the identity $\nabla^2 u = -A\mathbf{n}$ to obtain $A_{ij}A_{jk} - A_{il}A_{kj} = 0.\Box$

We conclude the proof of Theorem 1 as follows. Let u^k (resp. n^k) be the k^{th} component of $u \in I^{2,m}(\Omega,\mathbb{R}^{m+1})$ (resp. of $\mathbf{n} \in W^{1,m}(\Omega,\mathbb{R}^{m+1})$). We have $\nabla^2 u^k = -n^k A$. By Lemma 2.2 u^k satisfies the assumptions of Proposition 7. Hence $u^k \in C^1(\Omega)$. $\Box$

3 Developability

In this section we prove Theorem 2. As in the previous section, the main ingredient of the proof is the following result on the $W^{2,m}$ functions whose Hessian has rank less than or equal to 1.

Proposition 8. *Let $V \in W^{2,m}(\Omega)$ such that rank$(\nabla^2 V) \leq 1$ a.e. in Ω. Let*

$$\Omega_0 := \{x \in \Omega : \nabla V \textit{is constant in a neighborhood of } x\}.$$

Then, for any $x \in \Omega \setminus \Omega_0$, there exists a unique hyperplane of $\mathbb{R}^n$ such that ∇V is constant on the connected component of x in $P \cap \Omega$.

Proof: Let $f = \nabla V$. Let $Q \subset \Omega$ be an m-cube of the form $\Pi_{i=1}^m[a_i,b_i]$ such that $V \in W^{2,m}(K_i)$ for $1 \leq i \leq m$, where k_i is the i-dimensional skeleton of Q. First we claim that the proposition is true for the domain Q. Note that $Q = Q_2 \times Q_{m-2}$, where $Q_i \subset \mathbb{R}^i$ is an i-cube. By Fubini's theorem, for almost any $y \in Q_{m-2}$, $\nabla f^i \parallel \nabla f^1$ a.e. in $Q_2 \times \{y\}$, $g = (f^1,f^2)|_{Q_2\times\{y\}} \in W^{1,m}(Q_2 \times \{y\},\mathbb{R}^2)$ and g has singular symmetric gradient. Let

$$Q_0 := \{x \in Q_2 \times \{y\} : g \text{ is constant in a neighborhood of } x\}.$$

By [9], Proposition 1.1, for any $x \in Q_2 \times \{y\} \setminus Q_0$, g is constant on a segment l_x passing through x and joining the boundary of $Q_2 \times \{y\}$ at both ends. Therefore, by [2], Lemma 3, we conclude that for all $1 \leq i \leq m$, f^i is constant on the connected components of Q_0 and on all l_x for $x \in Q_2 \times \{y\} \setminus Q_0$. As a conclusion, and since f is continuous by Proposition 7, $f(Q) \subset f(\partial Q_2 \times Q_{m-2})$. Repeating the same argument, replacing Q by $m-1$ dimensional faces of $\partial Q_2 \times Q_{m-2}$, we conclude that $f(Q) \subset f(K_1)$, where K_1 is the collection of all 1-dimensional edges of Q. As a consequence, $S := f(Q)$

is of finite one dimensional Hausdorff measure. Since $f \in W^{1,m}(Q,S)$, we can apply the coarea formula to f, and we obtain that for almost $z \in S$, $f^{-1}(z)$ is an $(m-1)$-rectifiable current in Q. Let $x \in f^{-1}(z)$ for such z and let $\phi : [0,1] \to f^{-1}(z)$ be any Lipschitz curve with $\phi(0) = x$. We have for almost every $t \in [0,1]$,

$$0 = \nabla(f \circ \phi)(t) = \nabla f(\phi(t))\phi'(t), \tag{9}$$

therefore

$$\phi'(t) \in \mathrm{Ker}\nabla f(\phi(t)) = \mathrm{Im}(\nabla f(\phi(t))^{\perp} = (T_z S)^{\perp}.$$

Note that the use of the chain rule in (9) is justified by [10], Theorem 4.2. As a consequence we obtain that $\phi([0,1])$ lies in the plane P_x parallel to $(T_z S)^{\perp}$ passing through x. Since $f^{-1}(z)$ is $(m-1)$-rectifiable, using the continuity of f, we obtain that the connected component of $\{x\}$ in $P \cap Q$ is a subset of $f^{-1}(z)$ [3]. $\square$.

4 Proof of Theorem 4 and Corollary 6

Proof of Theorem 4: For S, an open domain in $\mathbb{S}^n$, let

$$\tilde{S} := \{rx \in \mathbb{R}^{n+1}; \quad x \in S, \; r \in [0,1]\}$$

and given $u \in I^{2,n}(S, \mathbb{S}^{n+1})$, consider the extension $\tilde{u} : \tilde{S} \to \mathbb{R}^{n+2}$

$$\tilde{u}(rx) = ru(x)$$

for all $x \in S$ and $r \in [0,1]$. A straightforward calculation shows that $\tilde{u}$ is an isometric immersion in $I^{2,n}(\tilde{S}, \mathbb{R}^{n+2})$. Following the final calculations in the proof of Theorem 1, all the 2-minors of the Hessian matrix of each component $\tilde{u}^k$ of $\tilde{u}$ vanish a.e. in $\tilde{S}$. Moreover, for all $1 \le i,j,k \le n$, $\nabla\partial_i\tilde{u}^k$ is a.e. parallel to $\nabla\partial_j\tilde{u}^k$.

It is sufficient to prove the corollary locally around any point $x \in S$. Without loss of generality we can assume that $x = e_{n+1} = (0, \cdots, 0, 1)$. By Fubini's theorem, there exists a hyperplane $P_0 \subset \mathbb{R}^{n+1}$, orthogonal to the direction e_{n+1} and with a distance $0 < \rho < 1$ from the origin such that the following properties hold:

(1) $\hat{u} := \tilde{u}|_{\tilde{S} \cap P_0} \in W^{2,n}(\tilde{S} \cap P, \mathbb{R}^{n+2})$,

(2) All the 2-minors of the Hessian matrix of each component $\hat{u}^k$ of $\hat{u}$ vanish a.e. in $\hat{S} := \tilde{S} \cap P_0$,

(3) For all $1 \leq i, j \leq n$, $1 \leq k \leq n+1$, $\nabla \partial_i \hat{u}^k$ is a.e. parallel to $\nabla \partial_j \hat{u}^k$ in $\hat{S}$.

Considering $\hat{u}$ as a function of the first n coordinates $(x_1, \cdots, x_n)$, and applying Proposition 7 to each component of $\hat{u}$ we conclude that $\hat{u} \in C^1(\hat{S})$. Moreover Proposition 8 and [2], Lemma 3 imply the existence of a subset $\hat{S}_0 \subset \hat{S}$ such that $\hat{u}$ is affine in $\hat{S}_0$ and for any point $x \in \hat{S}_0$, there exists a unique $n-1$ dimensional plane $P_1(x) \subset P_0$ such that $\hat{u}$ is affine on the connected component of $\hat{S} \cap P_1$. Notice that $\tilde{u}$ is a 1-homogeneous mapping, and therefore $\hat{u}(rx) = u(x)$ for all $rx \in \hat{S}$. A simple geometric observation completes the proof. $\Box$

Proof of Corollary 6: This is a straightforward consequence of Theorem 4 and the fact that any two copies of $\mathbb{S}^{n-1}$ in $\mathbb{S}^n$ intersect. $\Box$

5 Appendix

Here we will give another proof of Proposition 7 for even m. It would be interesting to see if this method can be adapted to the odd case. For $A \in M^{m \times m}$, we denote by $\widehat{A}_{ij}$ the $(m-1) \times (m-1)$ matrix obtained from A by eliminating the i^{th} row and the j^{th} column.

Lemma 5.1. *Let m be an even positive integer and let $B = [b_{ij}]_{m \times m}$ be the skew-symmetric matrix defined by $b_{ij} = 1 - \delta_{ij}$ for $i \geq j$. Then $\det B = 1$ and $\det \widehat{B}_{ij} + \det \widehat{B}_{ji} = 0$ for all $1 \leq i, j \leq m$. In particular $\det \widehat{B}_{ii} = 0$.*

Proof: We calculate $\det B$ by Gaussian elimination. Let R_i denote the i^{th} row of B.

$$\det B = \det \begin{bmatrix} R_1 \\ R_2 - R_3 \\ \vdots \\ R_{m-1} - R_m \\ R_m \end{bmatrix} = -\det \begin{bmatrix} R_m \\ R_2 - R_3 \\ \vdots \\ R_{m-1} - R_m \\ R_1 \end{bmatrix}$$

$$= -\det \begin{bmatrix} R_m \\ R_2 - R_3 \\ \vdots \\ R_{m-1} - R_m \\ R_1 - \sum_{i=2}^{\frac{1}{2}m-1} (R_{2i} - R_{2i+1}) \end{bmatrix} = \det \begin{bmatrix} -R_m \\ R_2 - R_3 \\ \vdots \\ R_{m-1} - R_m \\ \sum_{i=1}^{m-1} (-1)^{i+1} R_i \end{bmatrix}.$$

The last matrix is an upper triangular matrix whose diagonal entries are all equal to -1 since m is even. Hence $\det B = (-1)^m = 1$. On the other hand, $\widehat{B}_{ij} = -\widehat{B}_{ji}^T$ for all i, j. Hence $\det \widehat{B}_{ij} = (-1)^{m-1} \det \widehat{B}_{ji}$ which concludes the proof. □

Proof of Proposition 7 for even m**:** Write $f = \nabla V \in W^{1,m}(\Omega, \mathbb{R}^m)$ and $f_\delta = f + \delta b \in W^{1,m}(\Omega, \mathbb{R}^m)$, where $b : \Omega \to \mathbb{R}^m$ is the left multiplication by the skew symmetric matrix B introduced in Lemma 5.1. We have $\nabla f_\delta = \nabla f + \delta B$. Since all the minors of order ≥ 2 of $\nabla f = \nabla^2 V$ vanish, and since $A = [A_{ij}] = \nabla f$ is symmetric we obtain

$$\det(\nabla f_\delta) = \det(A + \delta B) = \delta^m \det B + \delta^{m-1} \left(\sum_{i,j=1}^{m} (-1)^{i+j} A_{ij} \det \widehat{B}_{ij} \right) = \delta^m > 0.$$

Hence f_δ is a deformation of $\Omega \subset \mathbb{R}^m$ in $\mathbb{R}^m$ with a.e. positive Jacobian. By [12] f_δ is open and continuous. Hence $f = \nabla V$ is continuous. □

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