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Convergence of Equilibria of Thin Elastic Plates – The Von Kármán Case

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We study the behaviour of thin elastic bodies of fixed cross-section and of height h, with $h \to 0$. We show that critical points of the energy functional of nonlinear three-dimensional elasticity converge to critical points of the von Kármán functional, provided that their energy per unit height is bounded by Ch^4 (and that the stored energy density function satisfies a technical growth condition). This extends recent convergence results for absolute minimizers.

Keywords Equilibria; Nonlinear elasticity; Plates; von Kármán equations.

Mathematics Subject Classification 74K20; 74B20.

1. Introduction and Main Result

The relation between three-dimensional nonlinear elasticity and theories for lower-dimensional objects such as rods, beams, membranes, plates and shells has been an outstanding question since the very beginning of the research in elasticity. Recently there has been substantial progress in the rigorous understanding of this relation through the use of variational methods, in particular Γ-convergence. This notion of convergence assures, roughly speaking, that absolute minimizers of the three-dimensional theory (subject to suitable boundary conditions and applied loads) converge to absolute minimizers of the limiting two-dimensional theory. In this paper we study the behaviour of possibly non-minimizing critical points of the energy functional. This is useful, e.g., to understand stability and bifurcation issues and might also be seen as a preliminary step towards a better understanding of the dynamic equations for which so far no rigorous results which start from the geometrically nonlinear three-dimensional theory are available. We focus on the scaling of applied forces and the elastic energy which leads to von Kármán's equations.

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To set the stage let us first review the variational setting. Consider a cylindrical domain $\Omega_h = S \times \left(-\frac{h}{2}, \frac{h}{2}\right)$, where S is a bounded subset of $\mathbb{R}^2$ with Lipschitz boundary. To a deformation $v : \Omega_h \to \mathbb{R}^3$ we associate the elastic energy (per unit height)

$$E^h(v) = \frac{1}{h} \int_{\Omega_h} W(\nabla v)\, dz. \tag{1.1}$$

We assume that the stored-energy density function W satisfies the following conditions:

$$W(RF) = W(F) \quad \forall R \in SO(3) \qquad \text{(frame indifference)}, \tag{1.2}$$

$$W = 0 \quad \text{on } SO(3), \tag{1.3}$$

$$W(F) \geq c \operatorname{dist}^2(F, SO(3)), \quad c > 0, \tag{1.4}$$

$$W \text{ is } C^2 \text{ in a neighbourhood of } SO(3). \tag{1.5}$$

Here $SO(3)$ denotes the group of proper rotations. The frame indifference implies that there exists a function $\widetilde{W}$ defined on symmetric matrices such that $W(\nabla v) = \widetilde{W}((\nabla v)^T \nabla v)$, i.e., the elastic energy depends only on the pull-back metric of v.

To discuss the limiting behaviour as $h \to 0$ it is convenient to rescale to a fixed domain $\Omega = S \times \left(-\frac{1}{2}, \frac{1}{2}\right)$ by the change of variables $z = (x_1, x_2, hx_3) = (x', hx_3)$ and $y(x) = (y'(x), y_3(x)) = v(z)$. With the notation

$$\nabla_h y = \left(\partial_1 y, \partial_2 y, \frac{1}{h}\partial_3 y\right) = \left(\nabla' y, \frac{1}{h}\partial_3 y\right) \tag{1.6}$$

we have

$$E^h(v) = I^h(y) = \int_\Omega W(\nabla_h y)\, dx. \tag{1.7}$$

The variational approach leads to a hierarchy of limiting theories depending on the scaling of I^h. More precisely we have for $h \to 0$ in the sense of Γ-convergence

$$\frac{1}{h^\beta} I^h \xrightarrow{\Gamma} I_\beta. \tag{1.8}$$

This implies, roughly speaking, that minimizers of I^h (subject to suitable boundary conditions or body forces) converge to minimizers of I_β, provided I^h evaluated on the minimizers is bounded by Ch^β. Such Γ-convergence was first established by LeDret and Raoult for $\beta = 0$ (see LeDret and Raoult, 1995), then for all $\beta \geq 2$ in Friesecke et al. (2002, 2006) (see also Pantz, 2001, 2003 for results for $\beta = 2$ under additional conditions). For $0 < \beta < 5/3$ convergence was recently obtained by Conti and Maggi (2008), see also Conti (2003). The exponent $\beta = 5/3$ is conjectured to be relevant for the crumpling of elastic sheets (see Conti and Maggi, 2008; Lobkovsky et al., 1995; Venkataramani, 2004).

Here we focus on the case $\beta = 4$ which leads to von Kármán's theory. For the limit problem we consider the averaged in-plane and out-of-plane displacements

$u \in W^{1,2}(S, \mathbb{R}^2)$ and $v \in W^{2,2}(S)$. An equilibrium is a critical point of the von Kármán functional

$$I^{vK}(u, v) = \frac{1}{2}\int_S Q_2\Big(\frac{1}{2}[\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v]\Big)\, dx' + \frac{1}{24}\int_S Q_2((\nabla')^2 v)\, dx' \quad (1.9)$$

subject to suitable boundary conditions (we will later include also applied vertical forces, see (1.12)). Here Q_2 is a quadratic form which can be computed from the linearization $\partial^2 W/\partial^2 F(Id)$ of the 3d energy at the identity. More precisely we consider the quadratic form

$$Q_3(F) := D^2 W(Id)F : F \quad (1.10)$$

and define the quadratic form $Q_2 : \mathbb{M}^{2\times 2} \to \mathbb{R}$ by minimizing Q_3 over stretches in the x_3 direction:

$$Q_2(G) = \mathcal{L}_2 G : G := \min_{F''=G} \{Q_3(F)\}. \quad (1.11)$$

The 2×2 submatrix F'' is given by $F''_{\alpha\beta} := F_{\alpha\beta}$ for $1 \le \alpha, \beta \le 2$. Note that if $W(F) = \frac{1}{2}\mathrm{dist}^2(F, SO(3))$, then simply $Q_2(A) = |\mathrm{sym}\, A|^2$. By

$$\mathrm{sym}\, A := \frac{A + A^T}{2}$$

we denote the symmetric part of a square matrix.

In this paper we study convergence of *equilibria* for the case $\beta = 4$. We consider the functional

$$J^h(y) = \int_\Omega W(\nabla_h y) - h^3 g(x') y_3\, dx\,, \quad (1.12)$$

subject to the clamped boundary conditions

$$y(x', x_3) = (x', hx_3) \quad \text{for all } x' \in \Gamma, \quad (1.13)$$

where Γ is a connected subset of ∂S of positive measure. The corresponding Γ-limit is given by

$$J^{vK}(\bar{u}, \bar{v}) = I^{vK}(\bar{u}, \bar{v}) - \int g\bar{v}dx'\,, \quad (1.14)$$

with the boundary conditions

$$\bar{u}(x') = 0, \quad \partial_\nu \bar{v}(x') = \bar{v}(x') = 0 \quad \text{for all } x' \in \Gamma, \quad (1.15)$$

where ν is the outward unit normal to ∂S. By calculating the respective variations of $J^{vK}(u, v)$ in v and in u we obtain the following Euler–Lagrange equations in weak form:

$$\int_S \Big(\frac{1}{2}\mathcal{L}_2(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v) : (\nabla' v \otimes \nabla' \phi) + \frac{1}{12}\mathcal{L}_2((\nabla')^2 v) :$$
$$(\nabla')^2 \phi - g\phi\Big) dx' = 0, \quad \forall \phi \in C^\infty(S), \quad \phi|_\Gamma = 0, \quad \nabla' \phi|_\Gamma = 0, \quad (1.16)$$

and

$$\int_S \mathscr{L}_2(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v) : \nabla' \psi \, dx' = 0, \quad \forall \psi \in C^\infty(S, \mathbb{R}^2), \ \psi|_\Gamma = 0, \quad (1.17)$$

with the boundary conditions

$$u(x') = 0, \quad \partial_\nu v(x') = v(x') = 0 \quad \text{for all } x' \in \Gamma. \quad (1.18)$$

We now assume in addition that $DW(F)$ grows at most linear at infinity, i.e.,

$$|DW(F)| \leq C(|F| + 1). \quad (1.19)$$

Together with the assumption that W is C^2 near the identity and is minimized at the identity this implies that

$$|DW(Id + A)| \leq C|A| \quad (1.20)$$

(with a different constant C).

Theorem 1.1. *Assume that* (1.2)–(1.5) *hold and that W is differentiable and satisfies the growth condition* (1.19). *Let* $(y^{(h)})$ *be a sequence of stationary points of J^h* (*subject to the boundary condition $y^{(h)}(x', x_3) = (x', hx_3)$ at $x' \in \Gamma$ and to natural boundary conditions on the remaining boundaries*). *Assume that*

$$\int_\Omega W(\nabla_h y^{(h)}) \leq Ch^4. \quad (1.21)$$

Let

$$U^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} \begin{pmatrix} y_1^{(h)} \\ y_2^{(h)} \end{pmatrix} (x', x_3) - x' dx_3, \quad V^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} y_3^{(h)} dx_3. \quad (1.22)$$

Then, up to subsequences,

$$v^{(h)} = \frac{1}{h} V^{(h)} \to v \quad \textit{in } W^{1,2}(S), \quad v \in W^{2,2}(S) \quad (1.23)$$

and

$$u^{(h)} = \frac{1}{h^2} U^{(h)} \rightharpoonup u \quad \textit{in } W^{1,2}(S) \quad (1.24)$$

as $h \to 0$, and the limit displacements (u, v) solve (1.16), (1.17) *and satisfy the boundary conditions* (1.18).

Remark 1.2. A careful application of the Poincaré inequality shows that the estimate (1.21) holds automatically for minimizers (see Friesecke et al., 2006, p. 219; Lecumberry and Müller, preprint, Lemma 13 for the details).

The growth condition on DW is unsatisfactory since it rules out blow-up of $W(F)$ as $\det F \to 0$ (which corresponds to a very strong compression). This is a well-known difficulty in nonlinear elasticity if one wants to go beyond absolute minimizers. Indeed, without such a growth condition it is not even known whether minimizers satisfy the Euler–Lagrange equations in the conventional form (3.1) given below (Ball, 2002). A condition which is compatible with blow-up for $\det F \to 0$ is $|F^T DW(F)| \leq C(W(F)+1)$. Ball has shown that under this condition one can obtain a variant of the Euler–Lagrange equation which involves the (weak) divergence of the energy-momentum tensor (rather than the stress itself) (Ball, 1983, 2002). It would be interesting to know whether our analysis can be extended to this setting.

In order to put the result above in context, one should mention the recent results of Mora et al. (2007) and of Mora and Müller (preprint) on the convergence of equilibria of respectively two and three dimensional thin elastic beams. Also it should be compared with a very interesting recent theorem of Monneau (2003). Monneau starts from a sufficiently smooth (and sufficiently small) solution of the von Kármán equations and he shows that there exists a nearby solution of the three-dimensional problem.

Remark 1.3. For the convenience of the reader, we would like to compare the equations (1.16)–(1.18) to what is known as the von Kármán equations in the literature. We should restrict ourselves to the case when the stored-energy function W is isotropic (i.e., $W(F) = W(FQ)$ for all $Q \in SO(n)$). In this case, the second derivative of W at the identity is

$$D^2W(Id)(F,F) = 2\mu|\mathrm{sym}\,F|^2 + \lambda|\mathrm{tr}(\mathrm{sym}\,F)|^2$$

where $\mu > 0$ and $\lambda \geq 0$ are the so called Lamé constants. A straightforward calculation yields

$$\mathscr{L}_2(F) = 2\mu\,\mathrm{sym}\,F + \frac{2\mu\lambda}{2\mu+\lambda}\mathrm{tr}(F)Id. \tag{1.25}$$

In what follows we assume that a solution (u, v) to (1.16)–(1.18) is smooth enough for our intentions. We also drop the prime symbol, keeping in mind that all the calculations are done in the 2d domain S. We have

$$\mathscr{L}_2(\nabla^2 v) = 2\mu\nabla^2 v + \frac{2\mu\lambda}{2\mu+\lambda}(\Delta v)Id.$$

Letting

$$N = \frac{1}{2}\mathscr{L}_2\big(\nabla u + (\nabla u)^T + \nabla v \otimes \nabla v\big),$$

we deduce the following equation satisfied in S from (1.17) and (1.18):

$$\mathrm{div}\,N = 0, \tag{1.26}$$

with the boundary conditions on Γ:

$$u = 0, \tag{1.27}$$

and the natural boundary conditions on $\partial S \setminus \Gamma$

$$N\nu = 0. \tag{1.28}$$

Similarly from (1.16) and (1.18) we have

$$\frac{\mu(\mu+\lambda)}{3(2\mu+\lambda)}\Delta^2 v - N : \nabla^2 v = g, \tag{1.29}$$

alongside with the boundary conditions on Γ:

$$v = \partial_\nu v = 0. \tag{1.30}$$

The natural boundary condition for v on $\partial S \setminus \Gamma$

$$\int_{\partial S} \nabla\phi \mathscr{L}_2(\nabla^2 v)\nu - \phi(\operatorname{div} \mathscr{L}_2(\nabla^2 v))\nu = 0, \quad \forall \phi \in C_c^\infty(\partial S \setminus \Gamma)$$

can be expressed as two second and third order point-wise conditions in terms of expressions involving ν and the unit tangent vector field τ to ∂S:

$$\nu \mathscr{L}_2(\nabla^2 v)\nu = 0, \quad 2\mu\partial_\tau(\nu(\nabla^2 v)\tau) + \frac{\mu(\mu+\lambda)}{3(2\mu+\lambda)}\partial_\nu \Delta v = 0, \quad \text{on } \partial S \setminus \Gamma. \tag{1.31}$$

The von Kármán equations are usually formulated in terms of *the Airy stress potential*. Note that N takes values in 2×2 symmetric matrices and satisfies (1.26) in S. Hence, if S is simply connected, there exists a function $A : S \to \mathbb{R}$ such that

$$N = \operatorname{cof}(\nabla^2 A).$$

A is the so called Airy stress function. Note that since A is determined modulo addition of an affine mapping, we can assume that $A(x_0) = \partial_\nu A(x_0) = 0$ for some $x_0 \in \partial S \setminus \Gamma$. Now, taking the divergence of (1.26) and taking into account (1.25), we obtain

$$\frac{4\mu(\mu+\lambda)}{2\mu+\lambda}(2\Delta \operatorname{div} u + \Delta|\nabla v|^2) + 4\mu \det \nabla^2 v = 0,$$

where we used the identity

$$\operatorname{div}^T \operatorname{div}(\nabla v \otimes \nabla v) = \Delta|\nabla v|^2 + 2\det \nabla^2 v.$$

Hence we have

$$\Delta^2 A = \Delta \operatorname{tr}(N) = \frac{\mu(2\mu+3\lambda)}{2\mu+\lambda}(2\Delta \operatorname{div} u + \Delta|\nabla v|^2) = -\frac{\mu(2\mu+3\lambda)}{\mu+\lambda}\det \nabla^2 v.$$

Introducing *the Monge–Ampère form*

$$[\phi, \psi] := \nabla^2\phi : \operatorname{cof}(\nabla^2\psi),$$

we note that (1.26) and (1.29) can be expressed as the following semilinear elliptic system of 4th order equations

$$\begin{cases} \frac{\mu(\mu+\lambda)}{3(2\mu+\lambda)}\Delta^2 v - [v, A] = g, \\ \Delta^2 A + \frac{\mu(2\mu+3\lambda)}{2(\mu+\lambda)}[v, v] = 0, \end{cases} \tag{1.32}$$

which are the equations proposed by von Kármán (1910). A straightforward observation shows that, if $\partial S \backslash \Gamma$ is connected, the boundary condition (1.28) can be expressed as

$$A = \partial_\nu A = 0 \quad \text{on } \partial S \backslash \Gamma. \tag{1.33}$$

Unfortunately the boundary conditions $u = 0$ on Γ cannot be translated into a point-wise condition for A.

In the literature, the system (1.32) is usually considered alongside the boundary conditions $A = \partial_\nu A = 0$ (1.33) and $v = \partial_\nu v = 0$ (1.30). In our setting, this is equivalent to letting $v = \partial_\nu v = 0$ and $Nv = 0$ on the whole boundary ∂S. Note that then u is determined only up to addition of an infinitesimal rigid motion $\binom{x_1}{x_2} \to \omega\binom{-x_2}{x_1} + a$ for constants $\omega, a \in \mathbb{R}$. It can be uniquely identified by imposing the constraints $\int_S u = 0$ and $\int_S u_1 x_2 - u_2 x_1 = 0$. See Ciarlet (1997), Chapter 5, for more discussion of this subject.

2. Properties of Low Energy Deformations

One difficulty in deriving the limiting theory is that smallness of the energy does not immediately imply that the gradient is close to the identity with the obvious scaling h^2 since frame indifference in principle allows for large rotations. A key ingredient of argument is the decomposition of the deformation gradient into a rotation (of order 1) which only depends on the in-plane variables and a strain of order h^2. At this point we use the rigidity estimates of Friesecke et al. (2002, 2006) which provide control of the rotation in terms of the energy.

Proposition 2.1 (Friesecke et al., 2006, Theorem 6 and Remark 5). *Let* $(y^{(h)}) \subset W^{1,2}(\Omega; \mathbb{R}^2)$ *be a sequence such that*

$$F^{(h)}(y^{(h)}) := \int_\Omega \operatorname{dist}^2\big(\nabla_h y^{(h)}, SO(3)\big)\, dx \le Ch^4,$$

for every $h > 0$. *Then there exists an associated sequence* $(R^{(h)}) \subset C^\infty(S; \mathbb{M}^{3\times3})$ *such that*

$$R^{(h)}(x') \in SO(3) \quad \textit{for every } x' \in S, \tag{2.1}$$

$$\|\nabla_h y^{(h)} - R^{(h)}\|_{L^2} \le Ch^2, \tag{2.2}$$

$$\|\nabla' R^{(h)}\|_{L^2} + h\,\|(\nabla')^2 R^{(h)}\|_{L^2} \le Ch \tag{2.3}$$

for every $h > 0$.

Proposition 2.2 (Friesecke et al., 2006, Lemma 1; Lecumberry and Müller, preprint, Lemma 13). *Let $y^{(h)}$ be as above. Assume that in addition*

$$y^{(h)}(x', x_3) = (x', hx_3) \quad \forall x' \in \Gamma,$$

where Γ is a connected subset of ∂S of positive measure, then

$$\|R^{(h)} - Id\,\|_{H^1} \leq Ch. \tag{2.4}$$

Set

$$U^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} \begin{pmatrix} y_1^{(h)} \\ y_2^{(h)} \end{pmatrix} (x', x_3) - x' dx_3, \quad V^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} y_3^{(h)} dx_3. \tag{2.5}$$

Then

$$v^{(h)} = \frac{1}{h} V^{(h)} \to v \in W^{2,2}(S) \;\; in \; W^{1,2}(S) \tag{2.6}$$

and

$$u^{(h)} = \frac{1}{h^2} U^{(h)} \rightharpoonup u \;\; in \; W^{1,2}(S) \tag{2.7}$$

as $h \to 0$. Moreover

$$\frac{R^{(h)} - Id}{h} \rightharpoonup A = -\nabla' v \otimes e_3 + e_3 \otimes \nabla' v \;\; in \; W^{1,2}(S), \tag{2.8}$$

and

$$\mathrm{sym}\frac{R^{(h)} - Id}{h^2} \to \frac{A^2}{2} = -\frac{1}{2}(\nabla' v \otimes \nabla' v + |\nabla' v|^2 e_3 \otimes e_3) \;\; in \; L^q(S), \;\; \forall q < \infty. \tag{2.9}$$

Finally, if $\zeta^{(h)}$ is the first moment of the displacement

$$\zeta^{(h)}(x') := \int_{\frac{1}{2}}^{\frac{1}{2}} x_3 \left[y^{(h)}(x', x_3) - \begin{pmatrix} x' \\ hx_3 \end{pmatrix} \right] dx_3.$$

we have

$$\frac{1}{h^2} \zeta^{(h)} \rightharpoonup \frac{1}{12} A e_3 = -\frac{1}{12} \begin{pmatrix} \nabla' v \\ 0 \end{pmatrix} \;\; in \; W^{1,2}(S, \mathbb{R}^3). \tag{2.10}$$

We also make use of the following well-known fact about weak convergence and linearization.

Proposition 2.3. *Let $1 \leq p < \infty$, let E be a bounded and measurable set in $\mathbb{R}^n$ and let $f : \mathbb{R}^d \to \mathbb{R}$ be a function which is differentiable at zero and satisfies $|f(a)| \leq M|a|$ for all $a \in \mathbb{R}^d$. Suppose that*

$$z^\delta \rightharpoonup z \;\; in \; L^p(E).$$

Then

$$\frac{1}{\delta} f(\delta z^{\delta}) \rightharpoonup Df(0)z \quad \text{in } L^p(E). \tag{2.11}$$

Proof. Since weak convergence commutes with the application of linear functions we may assume without loss of generality that $Df(0) = 0$. Set

$$\omega(\delta) := \sup_{|a| \leq \sqrt{\delta}} \frac{|f(a)|}{|a|}.$$

By assumption $\omega(\delta) \to 0$ as $\delta \to 0$. Set $A_\delta := \{x \in E : |z^\delta| \geq \delta^{-1/2}\}$. Then $|A_\delta| \to 0$ as $\delta \to 0$. Now assume first that $p > 1$. For every $g \in L^q(E, \mathbb{R}^d)$ (with $1/q + 1/p = 1$) we have

$$\left| \int_E g \cdot \frac{1}{\delta} f(\delta z^\delta)\, dx \right| \leq \omega(\delta) \|g\|_{L^q} \sup_\delta \|z^\delta\|_{L^p} + M \left(\int_{A_\delta} |g|^q\, dx \right)^{1/q} \sup_\delta \|z^\delta\|_{L^p}.$$

Since the right hand side converges to zero as $\delta \to 0$ we conclude that $\frac{1}{\delta} f(\delta z^\delta) \rightharpoonup 0$ in L^p as claimed. For $p = 1$ we use that fact that weak convergence in L^1 implies equiintegrability. For any g with $\|g\|_\infty \leq 1$ we have

$$\left| \int_E g \cdot \frac{1}{\delta} f(\delta z^\delta)\, dx \right| \leq C\omega(\delta) + M \int_{A_\delta} |z^\delta|\, dx$$

and the last term converges to zero by equiintegrability. This finishes the proof. □

3. Proof of Theorem 1.1

Proof. Let $(y^{(h)})$ be a sequence of stationary points of J^h, i.e., suppose that:

$$\int_\Omega (DW(\nabla_h y^{(h)}) : \nabla_h \varphi - h^3 g(x')\varphi_3)\, dx = 0 \tag{3.1}$$

for every $\varphi \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi = 0$ on $\Gamma \times (-\frac{1}{2}, \frac{1}{2})$. Assume that (1.21) holds true.

Step 1. Decomposition of the deformation gradients in rotation and strain.

By Propositions 2.1 and 2.2 there exists a sequence $(R^{(h)}) \subset C^\infty(S; \mathbb{M}^{3\times3})$ such that $R^{(h)}(x') \in SO(3)$ for every $x' \in S$ and

$$\|\nabla_h y^{(h)} - R^{(h)}\|_{L^2} \leq Ch^2, \tag{3.2}$$

$$\|\nabla' R^{(h)}\|_{L^2} + h\|(\nabla')^2 R^{(h)}\|_{L^2} \leq Ch, \tag{3.3}$$

$$\|R^{(h)} - \mathrm{Id}\|_{H^1} \leq Ch. \tag{3.4}$$

The estimates (3.2) and (3.4) imply that

$$\nabla_h y^{(h)} \to \mathrm{Id} \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{2\times2}).$$

In particular $\partial_3 y^{(h)} \to 0$ and thus

$$\nabla y^{(h)} \to \mathrm{diag}\,\{1, 1, 0\} \quad \text{strongly in } L^2(\Omega; \mathbb{M}^{2\times2}). \tag{3.5}$$

Since $|y^{(h)}(x', x_3)| \leq h$ for $x' \in \Gamma$, we deduce from Poincaré's inequality and (3.5) that $y^{(h)} \to (x', 0)$ strongly in $W^{1,2}(\Omega; \mathbb{R}^3)$. Assertions (1.23) and (1.24) follow from Proposition 2.2.

Now we make use of the approximated sequence of rotations $R^{(h)}$ to decompose the deformation gradients as

$$\nabla_h y^{(h)} = R^{(h)}\big(Id + h^2 G^{(h)}\big), \tag{3.6}$$

where the $G^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ are bounded in $L^2(\Omega; \mathbb{M}^{3\times 3})$ by (3.2). Thus, up to extracting a subsequence, we can assume that

$$G^{(h)} \rightharpoonup G \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}) \tag{3.7}$$

for some $G \in L^2(\Omega; \mathbb{M}^{3\times 3})$.

Step 2. Characterization of the limiting strain.

Proposition 3.1 (Friesecke et al., 2006, Lemma 2). *The 2×2 submatrix G'' given by $G''_{\alpha\beta} := G_{\alpha\beta}$ for $1 \leq \alpha, \beta \leq 2$ satisfies*

$$G''(x', x_3) = G_0(x') + x_3 G_1(x') \tag{3.8}$$

where

$$G_1 = -(\nabla')^2 v. \tag{3.9}$$

Moreover

$$\operatorname{sym} G_0 = \frac{1}{2}\big(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v\big). \tag{3.10}$$

Step 3. Consequences of the Euler–Lagrange equations.
Let $E^{(h)} : \Omega \to \mathbb{M}^{3\times 3}$ be the scaled stress defined by

$$E^{(h)} := \frac{1}{h^2} DW\big(Id + h^2 G^{(h)}\big). \tag{3.11}$$

In view of (1.20) and the L^2 bound on $G^{(h)}$ the functions $E^{(h)}$ are bounded in $L^2(\Omega; \mathbb{M}^{3\times 3})$, too. Moreover, by Proposition 2.3,

$$E^{(h)} \rightharpoonup E := \mathscr{L} G \quad \text{weakly in } L^2(\Omega; \mathbb{M}^{3\times 3}), \tag{3.12}$$

where the linear map $\mathscr{L}$ on matrix space is given by $\mathscr{L} := D^2 W(Id)$. This implies that E is symmetric. Indeed, let H be a skew-symmetric matrix. Then by frame indifference and the fact that $DW(Id) = 0$ we have $\mathscr{L} H = 0$ and therefore $(\mathscr{L} G, H) = (G, \mathscr{L} H) = 0$. This shows that E is symmetric.

By the decomposition (3.6) and by frame indifference we obtain that

$$DW(\nabla_h y^{(h)}) = R^{(h)} DW\big(Id + h^2 G^{(h)}\big) = h^2 R^{(h)} E^{(h)},$$

so that the Euler–Lagrange equations (3.1) can be written in terms of the stresses $E^{(h)}$ as

$$\int_\Omega \left(R^{(h)}E^{(h)} : \nabla_h \varphi - hg(x')\varphi_3\right) dx = 0 \tag{3.13}$$

for every $\varphi \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi = 0$ on $\Gamma \times \left(-\frac{1}{2}, \frac{1}{2}\right)$. Multiplying (3.13) by h and passing to the limit as $h \to 0$, we get

$$\int_\Omega Ee_3 \cdot \partial_3 \varphi \, dx = 0 \tag{3.14}$$

for every $\varphi \in W^{1,2}(\Omega; \mathbb{R}^3)$ with $\varphi = 0$ on $\Gamma \times (-\frac{1}{2}, \frac{1}{2})$. This yields $Ee_3 = 0$ a.e. in Ω. Since E is symmetric, we conclude that

$$E = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{12} & E_{22} & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{3.15}$$

Proposition 3.2. $E'' = \mathscr{L}_2 G''$.

Proof. We write $X = \mathbb{M}^{3\times 3}$ as the direct sum of two orthogonal subspaces $Y \oplus Y^\perp$ where $Y := \{F \in \mathbb{M}^{3\times 3} : F_{3i} = F_{i3} = 0, 1 \le i \le 3\}$. By $\pi : \mathbb{M}^{3\times 3} \to Y$ and $\pi^\perp : \mathbb{M}^{3\times 3} \to Y^\perp$ we denote the orthogonal projections to Y and $Y^\perp$, respectively. We need to show the following implication:

$$\mathscr{L}a \in Y \Longrightarrow \mathscr{L}a = \mathscr{L}_2 \pi a. \tag{3.16}$$

By definition of Q_2, for each $y \in Y$ there exists a $z \in Y^\perp$ such that $Q_2(y) = Q_3(y+z)$, and this z is characterized by

$$(\mathscr{L}(y+z), \zeta) = 0, \quad \text{for all } \zeta \in Y^\perp \tag{3.17}$$

(in fact z is unique up an irrelevant skew-symmetric matrix, since $\mathscr{L}$ is positive definite on symmetric matrices). By the linearity of this condition we have $Q_2(y_1 + y_2) = Q_3(y_1 + z_1 + y_2 + z_2)$, if each z_i satisfies (3.17) for y_i. Expanding both sides we see that $(\mathscr{L}_2 y_1, y_2) = (\mathscr{L}(y_1 + z_1), y_2 + z_2) = (\mathscr{L}(y_1 + z_1), y_2)$, where we used (3.17) in the last step. This yields $\mathscr{L}_2 y_1 = \mathscr{L}(y_1 + z_1)$. Now suppose that $\mathscr{L}a = 0$ and take $y_1 = \pi a$. Then $z_1 = \pi^\perp a$ satisfies (3.17) and $y_1 + z_1 = a$. This proves (3.16). □

Step 4. Symmetry properties of $E^{(h)}$.

Since W is frame indifferent, $W(\exp(tH)F) = W(F)$ for all skew-symmetric matrix $H \in \mathbb{M}^{3\times 3}$. Taking the derivative in t and letting $t = 0$ we obtain

$$DW(F)F^T : H = DW(F) : HF = 0.$$

Hence the matrix $DW(F)F^T$ is symmetric. Applying this with $F = Id + h^2 G^{(h)}$, we deduce that

$$E^{(h)} - (E^{(h)})^T = -h^2\left(E^{(h)}(G^{(h)})^T - G^{(h)}(E^{(h)})^T\right), \tag{3.18}$$

so that, using the boundedness of $E^{(h)}$ and $G^{(h)}$ in $L^2(\Omega; \mathbb{M}^{3\times 3})$, we have in particular the estimate

$$\|E^{(h)} - (E^{(h)})^T\|_{L^1} \leq Ch^2. \tag{3.19}$$

Step 5. Zeroth moment of the Euler–Lagrange equations.
We introduce the zeroth moment of the stress $E^{(h)}$, defined by

$$\overline{E}^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} E^{(h)}(x)\, dx_3,$$

for every $x' \in S$. We shall derive the Euler–Lagrange equations satisfied by the zeroth moment.

Let $\varphi = (\psi, \phi) : S \to \mathbb{R}^3$, $\psi \in C^\infty \cap W^{1,2}(S; \mathbb{R}^2)$ and $\phi \in C^\infty \cap W^{1,2}(S)$ be such that $\varphi(x') = 0$ for all $x' \in \Gamma$. Using the test function $\tilde{\varphi}(x', x_3) := \varphi(x')$ in the Euler–Lagrange equation (3.13) we obtain

$$\int_S \left(R^{(h)}\overline{E}^{(h)} : (\nabla'\varphi, 0) - hg\phi\right) dx' = 0. \tag{3.20}$$

Since $R^{(h)}$ is bounded and converges to Id in $W^{1,2}$ and since $\overline{E}^{(h)}$ weakly converges in L^2, we obtain in view of (3.14)

$$\int_S \overline{E}'' : \nabla'\psi\, dx' = 0, \quad \forall \psi \in W^{1,2}(S, \mathbb{R}^2), \quad \psi|_\Gamma = 0. \tag{3.21}$$

On the other hand, taking $\psi = 0$ and dividing (3.20) by h we obtain

$$\int_S \left(\frac{1}{h}\sum_{i=1}^{2}\left(\overline{E}_{3i}^{(h)}\partial_i\phi\right) + \sum_{i=1}^{2}\left(A^{(h)}\overline{E}^{(h)}\right)_{3i}\partial_i\phi - g\phi\right)dx' = 0, \tag{3.22}$$

for all $\phi \in C^\infty \cap W^{1,2}(S)$ such that $\phi|_\Gamma = 0$, where

$$A^{(h)} = \frac{R^{(h)} - Id}{h}.$$

By (2.8), $A^{(h)} \rightharpoonup A$ in $W^{1,2}(S, \mathbb{M}^{3\times 3})$ and hence $A^{(h)} \to A$ in all L^q, $2 \leq q < \infty$. As a consequence, $(A^{(h)}\overline{E}^{(h)})_{3i}$ is bounded in L^s for any $1 < s < 2$, and hence converges weakly in L^s to $(A\overline{E})_{3i}$. Recall that $Ee_3 = 0$ and that E is symmetric. Thus (2.8) yields

$$\left(A^{(h)}\overline{E}^{(h)}\right)_{3i} \rightharpoonup \sum_{j=1}^{2}(\partial_j v)\overline{E}_{ji} \quad \text{in } L^s.$$

Passing to the limit in (3.22) we obtain that

$$\int_S \frac{1}{h}\left(\overline{E}^{(h)}\right)^T e_3 \cdot (\nabla'\phi, 0)\, dx' \to \int_S \left(-\overline{E}'' : (\nabla' v \otimes \nabla'\phi) + g\phi\right) dx' \tag{3.23}$$

for all $\phi \in C^\infty \cap W^{1,2}(S)$ such that $\phi|_\Gamma = 0$.

Step 6. First moment of the Euler–Lagrange equations.

Let us also introduce the first moment of the stress $E^{(h)}$, defined by

$$\widehat{E}^{(h)}(x') := \int_{-\frac{1}{2}}^{\frac{1}{2}} x_3 E^{(h)}(x)\, dx_3,$$

for every $x' \in S$.

In order to obtain the Euler–Lagrange equation for the first moment, we set

$$\varphi(x', x_3) = x_3 R^{(h)} \begin{pmatrix} \eta(x') \\ 0 \end{pmatrix},$$

where $\eta \in C^\infty \cap W^{1,2} \cap L^\infty(S, \mathbb{R}^2)$ and $\eta|_\Gamma = 0$ and we apply (3.13). This yields

$$\int_\Omega \sum_{i=1}^{2} R^{(h)} x_3 E^{(h)} e_i \cdot \partial_i \left[R^{(h)} \begin{pmatrix} \eta \\ 0 \end{pmatrix} \right] + \frac{1}{h} R^{(h)} E^{(h)} e_3 \cdot R^{(h)} \begin{pmatrix} \eta \\ 0 \end{pmatrix} dx = 0,$$

and thus

$$\int_S \widehat{E}''^{(h)} : \nabla' \eta + \sum_{i=1}^{2} R^{(h)} \widehat{E}^{(h)} e_i : \partial_i R^{(h)} \begin{pmatrix} \eta \\ 0 \end{pmatrix} + \frac{1}{h} \overline{E}^{(h)} e_3 \cdot \begin{pmatrix} \eta \\ 0 \end{pmatrix} dx' = 0. \tag{3.24}$$

Note that $R^{(h)}$ is uniformly bounded in L^∞ and $R^{(h)} \to Id$ in $W^{1,2}(S)$. Thus taking $\eta = \nabla' \phi$ we conclude that

$$\int_S \frac{1}{h} \overline{E}^{(h)} e_3 \cdot (\nabla' \phi, 0)\, dx' \to -\int_S \widehat{E}'' : (\nabla')^2 \phi\, dx', \tag{3.25}$$

for all $\phi \in C^\infty \cap W^{2,2}(S)$, $\nabla' \phi = 0$ on Γ.

Step 7. Derivation of the limit equations.

By (3.8)–(3.10) and Proposition 3.2 we obtain that

$$E'' = \mathcal{L}_2 G'' = \mathcal{L}_2 G_0 + x_3 \mathcal{L}_2 G_1.$$

As a consequence

$$\overline{E}'' = \mathcal{L}_2(\operatorname{sym} G_0) = \mathcal{L}_2\Big(\frac{1}{2}(\nabla' u + (\nabla' u)^T + \nabla' v \otimes \nabla' v)\Big). \tag{3.26}$$

Therefore (1.17) follows from (3.21). On the other hand

$$\widehat{E}'' = -\frac{1}{12} \mathcal{L}_2\big((\nabla')^2 v\big), \tag{3.27}$$

which combined with and (3.19), (3.23), (3.25) yields (1.16).

From the boundary conditions for $y^{(h)}$, we obtain immediately that $u(x') = 0$ and $v(x') = 0$ for all $x' \in \Gamma$. In order to conclude, we only need to show $\nabla' v = 0$ on Γ. To this end we observe that $\zeta^{(h)}(x') = 0$ for all $x' \in \Gamma$, where $\zeta^{(h)}$ is the first moment of the displacement as defined in Proposition 2.2. Together with the compact embedding from $W^{1,2}(S)$ to $L^2(\partial S)$ this proposition yields $\nabla' v = 0$ on Γ. This finishes the proof of Theorem 1.1. □

Remark 3.3. The main ingredient of the proof of the convergence theorems in Mora and Müller (preprint) is establishing the convergence of the energies (up to a subsequence)

$$\int_\Omega E^{(h)} : G^{(h)} \to \int_\Omega E : G = \int_\Omega \mathscr{L}G : G. \tag{3.28}$$

Here this is not needed for proving our result, however (3.28) is a straightforward Corollary of Theorem 1.1, see below. As a consequence, we can establish strong convergence of the symmetric parts of the strains, $\operatorname{sym} G^{(h)}$, and of the stresses $E^{(h)}$, assuming that $hG^{(h)}$ converges to 0 uniformly. If this assumption is not satisfied, one can introduce an auxiliary sequence of truncated deformations, whose corresponding scaled strains satisfy this condition. See Mora et al. (2007) and Mora and Müller (preprint) for more details.

To conclude the remark and for the convenience of the reader we give a proof of (3.28). First note that $z^{(h)} := y^{(h)} - (x', hx_3)$ can be used as a test function in the Euler–Lagrange equations (3.13). Hence we obtain

$$\begin{aligned}
\int_\Omega E^{(h)} : G^{(h)} &= \int_\Omega R^{(h)}E^{(h)} : R^{(h)}G^{(h)} \\
&= \int_\Omega R^{(h)}E^{(h)} : \left(\frac{Id - R^{(h)}}{h^2}\right) + \int_\Omega R^{(h)}E^{(h)} : \frac{1}{h^2}\nabla_h z^{(h)} \\
&= \int_\Omega \frac{1}{2}\left(E^{(h)} - (E^{(h)})^T\right) : (R^{(h)})^T\left(\frac{Id - R^{(h)}}{h^2}\right) \\
&\quad + \int_\Omega E^{(h)} : \operatorname{sym}\left(\frac{(R^{(h)})^T - Id}{h^2}\right) + \int_\Omega \frac{1}{h} g(x') z_3^{(h)}\, dx'\, dx_3.
\end{aligned} \tag{3.29}$$

Using (2.9) and (3.19) and applying the dominated convergence theorem we obtain that the first term on the right hand side converges to 0. Also by (2.9), (1.17) and (3.26) we have for the second term

$$\int_\Omega E^{(h)} : \operatorname{sym}\left(\frac{(R^{(h)})^T - Id}{h^2}\right) \to -\int_S \mathscr{L}_2 G_0 : \operatorname{sym} G_0.$$

On the other hand applying (2.6), (1.16) and (1.17) we have

$$\begin{aligned}
\int_\Omega \frac{1}{h} g(x') z_3^{(h)}\, dx'\, dx_3 &= \int_S g(x')\left(\int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{1}{h}(y_3^{(h)} - x_3)\, dx_3\right) dx' \\
&= \int_S g(x') \frac{1}{h} V^{(h)}\, dx' \to \int_S g(x') v(x')\, dx' \\
&= 2\int_S \mathscr{L}_2 G_0 : \operatorname{sym} G_0 + \frac{1}{12}\mathscr{L}_2 G_1 : G_1.
\end{aligned}$$

Therefore we finally obtain

$$\int_\Omega E^{(h)} : G^{(h)} \to \int_S \mathscr{L}_2 G_0 : G_0 + \frac{1}{12}\mathscr{L}_2 G_1 : G_1 = \int_\Omega \mathscr{L}_2 G'' : G'' = \int_\Omega E : G.$$

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