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Regularity properties of isometric immersions

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Abstract We show that an isometric immersion y from a two-dimensional domain S with $C^{1,\alpha}$ boundary to $\mathbb{R}^3$ which belongs to the critical Sobolev space $W^{2,2}$ is C^1 up to the boundary. More generally C^1 regularity up to the boundary holds for all scalar functions $V \in W^{2,2}(S)$ which satisfy $\det \nabla^2 V = 0$. If S has only Lipschitz boundary we show such V can be approximated in $W^{2,2}$ by functions $V_k \in W^{1,\infty} \cap W^{2,2}$ with $\det \nabla^2 V_k = 0$.

1. Introduction

In this paper we study isometric immersions y from a two-dimensional set S to $\mathbb{R}^3$ which are in the Sobolev class $W^{2,2}$, i.e. $\nabla^2 y$ is in L^2 (this is equivalent to the condition that the second fundamental form A is in L^2). The motivation to study this class arises on the one hand from geometry (where the class $W^{2,2}$ corresponds to an interesting borderline case) and on the other hand from nonlinear plate theory, where the $W^{2,2}$ isometric immersions form the natural class of admissible functions.

Regarding geometry it is well known that C^2 isometric immersions have a good classification and enjoy strong rigidity properties while the celebrated results of Nash [6] and Kuiper [11] show that C^1 isometric immersions can be much more complicated (e.g. the image of S^2 can be contained in an arbitrarily small ball). The class $W^{2,2}$ lies somewhat in between. We have information on second derivatives, but only in an integral sense (C^1 surfaces with even weaker properties have been studied by Pogorelov [16], [17, Chapter IX]). As we will see in this class the usual geometric properties still hold, in particular the image is a ruled surface (see Theorem 4 below). This is no longer true if we make the slightly weaker assumption that $\nabla^2 y$ is in L^p, for all $p < 2$. In that case a conical singularity can occur (consider e.g. polar coordinates $(x_1, x_2) = (r\cos\varphi, r\sin\varphi)$ and the one-homogeneous map $y(x) = (\frac{1}{2}r\cos 2\varphi, \frac{1}{2}r\sin 2\varphi, \frac{1}{2}\sqrt{3}r))$.

Regarding elasticity, the above class arises naturally in the geometrically nonlinear theory of plates, first formulated by Kirchhoff [10] and recently rigorously derived from three dimensional nonlinear elasticity by variational methods [2] (for a derivation under more restrictive hypotheses see [14, 15]). In this theory the admissible maps are isometric immersions and the energy is $\int_S Q_2(A)$ where A is the second fundamental form and Q_2 is a positive definite quadratic form. From this one easily sees that finite energy implies $y \in W^{2,2}$ (see [2], Remark (vii) after Theorem 6.1).

In general maps in $W^{2,2}$ just fail to be in C^1 (critical Sobolev embedding). This first main result asserts that isometric immersions are in C^1 *up to the boundary*, if S is sufficiently regular (for Lipschitz S the gradient may blow up at the boundary, see Remark 7 below).

Theorem 1. *Suppose that $\alpha > 0$ and that $S \subset \mathbb{R}^2$ is a bounded domain with $C^{1,\alpha}$ boundary, i.e. ∂S can be covered by finitely many charts in which ∂S is a $C^{1,\alpha}$ graph and within each chart S lies above that graph. Let $y \in W^{2,2}(S, \mathbb{R}^3)$ be an isometric immersion, i.e $(\nabla y)^T \nabla y = Id$ almost everywhere. Then y is C^1 up to the boundary, with a logarithmic modulus of continuity. More precisely there exists a constant (depending only on S) such that for $r < R/4 < R_0(S)$ and for every $x \in S$*

$$\begin{aligned} \operatorname{osc}_{B(x,r)\cap S} \nabla y &\leq C \ln^{-1/2}(R/r)||\nabla^2 y||_{L^2(B(x,R)\cap S)} \\ &= C \ln^{-1/2}(R/r)||A||_{L^2(B(x,R)\cap S)}. \end{aligned}$$

Here $\operatorname{osc}_{B(x,r)} f$ denotes the oscillation of f on a ball of radius r around x, i.e. the diameter of the image $f(B(x, r))$. To prove this estimate we use the fact that each component of y^k of y satisfies $\det \nabla^2 y^k = 0$ (this is classical for smooth isometric immersions, for the $W^{2,2}$ case see Proposition 3 below). We then establish the oscillation estimate for ∇V for all scalar functions $V \in W^{2,2}(S)$ which satisfy

$$\det \nabla^2 V = 0, \tag{1}$$

see Theorem 6 below. Equation (1), which is equivalent to the fact that the Gauss curvature of graph V vanishes, also plays an important role in the study of isometries which are close to the trivial map $x \mapsto (x, 0)$. To study them it is natural to consider the ansatz

$$y(x) = \begin{pmatrix} x + \delta^2 U(x) \\ \delta V(x) \end{pmatrix}, \quad \text{where } U : S \to \mathbb{R}^2,\ V : S \to \mathbb{R}. \tag{2}$$

The condition that y is an isometry becomes

$$\delta^2(\nabla U + (\nabla U)^T + \nabla V \otimes \nabla V) + \delta^4 (\nabla U)^T \nabla U = 0, \tag{3}$$

and it is natural to consider the formal linearization

$$\nabla U + (\nabla U)^T + \nabla V \otimes \nabla V = 0. \tag{4}$$

Suppose that $V \in W^{2,2}$ is given. Then one can easily check that (1) is a necessary and sufficient condition for the existence of a U satisfying the linearized relation (4).

If in addition $\delta|\nabla V| < 1$ then the same condition is necessary and sufficient for the existence of a U satisfying the full isometry condition (3) (see [4]).

Thus one is interested whether condition (1) already implies that V is Lipschitz. For $C^{1,\alpha}$ domains one even has C^1 regularity up to the boundary (see Theorem 6 below). For Lipschitz domains, however, ∇V may be unbounded, see Remark 7 below. We show that nonetheless ∇V can be approximated in $W^{2,2}$ by Lipschitz functions which still satisfy (1).

Theorem 2. *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $V \in W^{2,2}(S)$ satisfies*

$$\det \nabla^2 V = 0.$$

Then there exists an increasing sequence of open subsets S_k and maps $V_k \in W^{2,2}(S)$ such that

$$||\nabla V_k||_{L^\infty(S)} \leq k, \quad V_k = V \quad \text{in } S_k,$$

$$\nabla^2 V_k = 0 \quad \text{a.e. on } S \setminus S_k,$$

$$\bigcup_{k=1}^{\infty} S_k = S.$$

In particular we have $\det \nabla^2 V_k = 0$, $||\nabla V_k||_{L^2} \leq ||\nabla V||_{L^2}$ *and* $V_k \to V$ *in* $W^{2,2}(S)$.

Applications of these results to the derivation of plate theories from three dimensional nonlinear elasticity and to the stability analysis of plates are discussed in [4,5].

2. Properties of $W^{2,2}$ isometric immersions and solutions of $\det \nabla^2 v = 0$

Following [4] we first review some general properties of isometric immersions for the convenience of the reader. These properties are classical for smooth maps, but we will need them for $W^{2,2}$ maps. For a general $W^{2,2}$ map $y : S \to \mathbb{R}^3$ we define the induced metric by $g_{ij} = y_{,i} \cdot y_{,j}$ and we set $n = y_{,1} \wedge y_{,2}$ and

$$A_{ij} = -y_{,ij} \cdot n. \tag{5}$$

If y is an isometric immersion, i.e. if $g_{ij} = \delta_{ij}$, then n is the normal to the image of y and A is the second fundamental form.

Proposition 3. *Suppose that S is a bounded Lipschitz domain and $y \in W^{2,2}(S; \mathbb{R}^3)$ is an isometric immersion. Then*

$$y_{,ij} = -A_{ij} n, \tag{6}$$

$$A_{i1,2} = A_{i2,1}, \quad for \;\; i = 1, 2, \tag{7}$$

in the sense of distributions. Moreover

$$\det A = 0. \tag{8}$$

Proof. This follows from standard approximation arguments. Since $g_{ij} = \delta_{ij}$ we have $|n| = 1$. Differentiation of g_{ij} yields after a short calculation that $y_{,ij} \cdot y_{,k} = 0$ a.e. Thus $y_{,ij}$ is parallel to n and this proves (6). To establish (7) first note that for smooth y we have the identity

$$A_{i1,2} - A_{i2,1} = -y_{,i1} \cdot n_{,2} + y_{,i2} \cdot n_{,1} \tag{9}$$

By approximation this identity holds in the sense of distributions if $y \in W^{2,2}$. By (6) the vector $y_{,ij}$ is parallel to n (a.e.), but $n_{,k}$ is perpendicular to n, since $|n| = 1$. This proves (7).

Finally to establish (8) we start from the identity

$$g_{11,22} + g_{22,11} - 2g_{12,12} = 2y_{,12} \cdot y_{,12} - 2y_{,11} \cdot y_{,22}. \tag{10}$$

This holds pointwise for smooth y and by approximation it holds in the sense of distribution for $y \in W^{2,2}$. For an isometric immersion the left hand side vanishes and together with (6) this proves (8). $\square$

If y is smooth then one can deduce from (8) that locally the image of ∇y is either a constant or a smooth curve. In the latter case one can further conclude that ∇y is constant on lines defined by the kernel of A. For C^2 isometric immersions this assertion is contained in the more general results of Hartman and Nirenberg [7]. Pogorelov [16, Chapter II], [17, Chapter IX] has obtained the same result under very weak hypotheses. He only requires that the immersion is C^1 and that the image of the Gauss map has measure zero in S^2. A short proof under the stronger hypothesis that the isometric immersion is in $W^{2,2}$ was recently given by Pakzad [13], using an idea of Kirchheim [9]. For later use we state his result both for functions (with $\det \nabla^2 V = 0$) and for isometric immersions.

Theorem 4. *[13] Let S be a bounded Lipschitz domain. Suppose that $V \in W^{2,2}(S)$ with* $\det \nabla^2 V = 0$. *Consider the open set*

$$S_1 = \{x \in S \,:\, \nabla V \textit{ is constant in a neighbourhood of } x\}. \tag{11}$$

Then through every point $x \in S \setminus S_1$ there exists a line segment which intersects ∂S at both ends and on which ∇V is constant. Different line segments do not intersect in S.

The same characterization holds for an isometric immersion in $W^{2,2}(S; \mathbb{R}^3)$.

Even though a general function in $W^{2,2}(S)$ need not be C^1 for isometric immersions (and more generally for solutions of $\det \nabla^2 V = 0$) one can easily obtain an interior C^1 estimate.

Proposition 5. *Suppose that* $V \in W^{2,2}(S)$ *and* $\det \nabla^2 V = 0$. *Then* $V \in C^1(S)$. *If* $B_\rho(x) \subset B_R(x) \subset S$ *we have more precisely*

$$\operatorname{osc}_{B_\rho} \nabla V \leq C(\ln \frac{R}{\rho})^{-1/2} ||\nabla^2 V||_{L^2(B_R)}, \tag{12}$$

where $\operatorname{osc}_{B_\rho} f := \operatorname{diam} f(B_\rho)$.

Proof. Following Kirchheim we set $f = \nabla V$ and $g(x) = f(x) + \delta(-x_2, x_1)$ and g is in the borderline space $W^{1,2}$. By a result of Vodopyanov and Goldstein [19] g is therefore continuous (see also [18,8,1]). In fact g is a monotone map, i.e. $\operatorname{osc}_{B_r} g = \operatorname{osc}_{\partial B_r} g$, for all r. From this one easily sees that g and hence f have the asserted modulus of continuity (see [12], Thm. 4.3.4). We sketch the details for the convenience of the reader. By Fubini's theorem g belongs to $W^{1,2}(\partial B_r; \mathbb{R}^2)$ for almost every r. The Sobolev embedding theorem applied to the one-dimensional set ∂B_r yields, for a.e. $r \in (\rho, R)$

$$\operatorname{osc}_{B_\rho} g \leq \operatorname{osc}_{B_r} g = \operatorname{osc}_{\partial B_r} g \leq \sqrt{r} \left(\int_{\partial B_r} |\nabla g|^2 \right)^{1/2}.$$

Now take squares, divide by r and integrate from ρ to R. This gives the estimate for g and the one for f follows by letting δ go to zero. □

3. C^1 estimates up to the boundary

In this section we establish the following estimate.

Theorem 6. *Suppose that* $\alpha > 0$ *and* $S \subset \mathbb{R}^2$ *is a bounded domain with* $C^{1,\alpha}$ *boundary. Suppose* $V \in W^{2,2}(S)$ *satisfies*

$$\det \nabla^2 V = 0.$$

Then V *is* C^1 *up to the boundary and there exist constants* $R_0(S)$, $C(S)$ *(depending only on* S*) such that for* $r < R/4 < R_0(S)$ *and for every* $x \in S$

$$\operatorname{osc}_{B(x,r)\cap S} \nabla V \leq C(S) \ln^{-1/2}(R/r) ||\nabla^2 V||_{L^2(B(x,R)\cap S)}$$

This implies in particular Theorem 1 since by (6) and (8) each component y^k of an isometric immersion satisfies $\det \nabla^2 y^k = 0$. Moreover (6) also shows that $|\nabla^2 y| = |A|$.

Remark 7. The result does not hold for Lipschitz domains. Consider for example the truncated cone $\{(x_1, x_2) : x_1 \in (-1, 1), |x_1| < x_2 < 1\}$ and $V(x) = v(x_2)$ with $v'(0) = \infty$ and $\int_0^1 t|v''(t)|^2 < \infty$. One may take e.g. $v'(t) = |\ln t|^\alpha, 0 < \alpha < 1/2$. A slight modification shows that even C^1 domains are not sufficient. One needs a certain logarithmic modulus of continuity of the normal.

The proof of Theorem 6 uses the condition $\det \nabla^2 V = 0$ only to deduce the existence of the line segments which appear in Theorem 4. To stress this and in view of possible future applications we will in the following only use this condition. We say that a function $f : S \to \mathbb{R}^2$ satisfies **condition (L)** if the following holds:

> Let S_1 be the open set on which f is locally constant. Then through every point $x \in S \setminus S_1$ there exists a line segment which intersects ∂S at both ends and on which f is constant. Different line segments do not intersect in S.

To prove Theorem 6 we first use the condition (L) together with the Poincaré inequality to derive an oscillation bound on half-discs. The main point is then to establish the oscillation bound on line segments normal to the boundary. For this it suffices to consider domains whose boundary is a 'parabola' $x_2 = |x_1|^{1+\alpha}$ and we study these in Lemma 9. The heuristic idea is simple. Suppose for simplicity that through every point there is a line segment on which $f := \nabla V$ is constant. If these line segments intersect the boundary transversally at a point $\bar{x}$ then their length is bounded from below and one can apply the Poincaré inequality normal to the line segments to obtain an oscillation bound. Suppose now that the slope of the line segments approaches that of the tangent at $\bar{x}$. In the extreme case that all line segments are parallel to the tangent we are exactly in the situation of Remark 7, but now the $C^{1,\alpha}$ regularity of the boundary implies that the length of the segments scales like $x_2^{1/1+\alpha}$ where x_2 is the normal variable. Hence one obtains a less degenerate weight in the one-dimensional estimate and one easily obtains the oscillation estimate from the Cauchy-Schwarz inequality. The estimates (25)–(27) capture this fact in the general situation where the lines may not all be parallel to the tangent.

We begin with the oscillation estimate on half-discs $B^+(x, r) = \{y \in \mathbb{R}^2 : |y - x| < r, y_2 > x_2\}$.

Proposition 8. *Suppose that $f \in W^{1,2}(S, \mathbb{R}^2)$ has property (L) and that the closure of $B^+(x, R)$ is contained in S. Then for all $r < R/2$*

$$\operatorname{osc}_{B^+(x,r)} f \leq C \ln^{-1/2}\left(\frac{R}{r}\right) ||\nabla f||_{L^2(B^+(x,R))}. \tag{13}$$

Proof. We may suppose without loss of generality that the half-discs are centered at zero. Moreover it suffices to compare $f(0)$ to $f(y)$ with $|y| < r$. Suppose first that both through 0 and through y there exist a line segment on which f is constant and denote them by l_0 and l. In polar coordinates (ρ, φ) the first segment is given by $\varphi = \varphi_0$ while the part of the second segment which lies in the annulus $r < \rho < R$ can be decribed by a bounded function $\varphi = h(\rho)$. An application of the Poincaré inequality in polar coordinates yields

$$|f(y) - f(0)|^2 \leq C|h(\rho) - \varphi_0|\rho \int_{\partial B_\rho \cap B_R^+} |\nabla f|^2 \, d\mathcal{H}^1$$

Dividing by ρ and integrating over ρ from r to R we obtain (13).

If 0 or y belong to the set S_1 where f is locally constant consider the segment from 0 to y. If all points on this segment belong to S_1 then $f(0) = f(y)$. Otherwise let p and q be the points on the intersection of the segment and ∂S_1 which are closest 0 and y, respectively. Then $f(p) = f(0)$, $f(q) = f(y)$. Let $R' = R - |p|$, $r' = r - |p|$. Then $q \in B^+(p, r')$ and the closure of $B^+(p, R')$ is contained in S. Hence by the previous argument we obtain the desired bound with R/r replaced by R'/r'. Since the latter quantity is bigger than or equal to the former this finishes the proof. □

We next estimate the oscillation in normal direction. For this it suffices to consider a parabola shaped domain, see Fig. 1.

Lemma 9. *Consider the domain*

$$\Omega = \{(x_1, x_2) : x_1 \in (-2, 2),\ |x_1|^{1+\alpha} < x_2 < 2^{1+\alpha}\}.$$

Suppose that $f \in W^{1,2}(\Omega; \mathbb{R}^2)$ *has the property (L) (with respect to* Ω*) and consider*

$$F(t) = f(0, t).$$

Then

$$|F(t) - F(t')| \leq C \ln^{-1/2}\left(\frac{1}{|t - t'|}\right) ||\nabla f||_{L^2(\Omega)}, \quad \forall t, t' \in (0, 1). \tag{14}$$

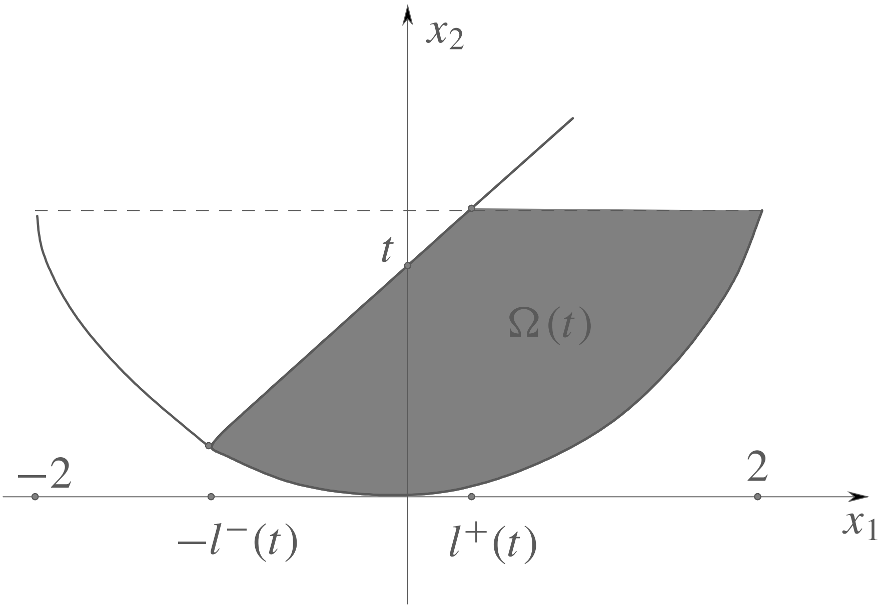

Fig. 1 The parabolic domain Ω and the domains $\Omega(t)$ generated by line segments through the points $(0, t)$

Proof. We first prove the result under the additional assumption that through every point $(0, t)$ (with $t \leq 1$) there is a line segment (touching $\partial\Omega$ at both ends) on which f is constant. Let $s(t)$ be the slope of this segment and let $-l^-(t)$ and $l^+(t)$ denote the x_1 coordinates of the left and right intersection point of the segment with $\partial\Omega$. Let further $\Omega(t)$ denote the area under the line segment (see Figure 1) and set

$$G(t) = \int_{\Omega(t)} |\nabla f|^2 \, dx.$$

We will show that F, G and s are absolutely continuous and satisfy, for a.e. $t \in (0, 1)$,

$$|\dot{F}| \leq |\dot{G}|^{1/2}(1 + s^2(t))^{-1/2} h^{-1/2}(t), \tag{15}$$

where

$$h(t) \geq \frac{1}{|\dot{s}(t)|} \ln(1 + |\dot{s}(t)|\bar{l}(t)) \tag{16}$$

and $\bar{l}(t) = \max(l^-(t), l^+(t))$. Together with simple geometric estimates on l^+ and l^- and a short calculation (see Proposition 10 below) this will imply (14).

Step 1. Estimates for $l^\pm$ and a $W^{1,1}$ estimate for $\beta = \arctan s$.

Note first that we can assume that the slope s is finite at each point $(0, t)$. Otherwise f is constant on the segment $t \mapsto (0, t)$ and there is nothing to show. Assume for the moment $s \geq 0$. If the line segment through $(0, t)$ does not intersect the upper boundary $x_2 = 2^{1+\alpha}$ then $l^\pm(t)$ are given by the equations

$$t + sl^+ = (l^+)^{1+\alpha}, \quad t - sl^- = (l^-)^{1+\alpha}.$$

Thus

$$\begin{aligned} l^+ &\geq t^{1/(1+\alpha)}, \\ l^- &\geq \min\left(t/(2s), (t/4)^{1/(1+\alpha)}\right). \end{aligned} \tag{17}$$

If the line segment does intersect the upper boundary $x_2 = 2^{1+\alpha}$ then $l^+ \geq 1/s$ (here and in the following we always assume $t \leq 1$). Hence we always have, for $s(t) \geq 0$,

$$l^+(t) \geq \min\left(s^{-1}, t^{1/(1+\alpha)}\right) \tag{18}$$

For $s(t) < 0$ the roles of l^+ and l^- are interchanged.

By property (L) different line segments do not intersect in Ω. From this we easily conclude that s is locally Lipschitz and

$$|\dot{s}| \leq \frac{1}{\min(l^-, l^+)}.$$

To obtain better integral estimates for $\dot{s}$ we first show that $|s|$ is almost increasing in t. Suppose again that at the point t we have $s(t) < 0$. Then the upper derivative

$$\dot{s}_+(t) := \limsup_{\tau \to 0} \max\left(\frac{s(t + \tau) - s(t)}{\tau}, 0\right)$$

satisfies

$$\dot{s}_+ \le \frac{1}{l^-} \le \max(t^{-1/1+\alpha}, s).$$

Combining this with an analogous estimate for $\dot{s}_-$ if $s(t) \ge 0$ we get

$$-\frac{d}{dt}|s| \le \max(t^{-\frac{1}{1+\alpha}}, |s|). \tag{19}$$

From this we easily deduce that the function $t \mapsto e^t(|s|(t) + \frac{1+\alpha}{\alpha}t^{\alpha/1+\alpha})$ is increasing and we obtain a $W^{1,1}$ bound for this function (and hence for $|s|$ and s) in terms of $s(1)$, the slope at $t = 1$. This slope, however, cannot be controlled in terms of ∇f alone and it is therefore more convenient to work with the angle

$$\beta = \arctan s$$

instead of the slope. From (19) we get

$$-\frac{d}{dt}|\beta| = -\frac{1}{1+s^2}\frac{d}{dt}|s| \le t^{-\frac{1}{1+\alpha}}, \quad \text{for } t \in (0, 1).$$

Thus $\frac{d}{dt}|\beta| \ge -t^{-1/(1+\alpha)}$ and hence $\sigma = |\beta| + \frac{1+\alpha}{\alpha}t^{\alpha/(1+\alpha)}$ is monotone. Since β takes values in $(-\pi/2, \pi/2)$ we get

$$\int_\varepsilon^{1-\varepsilon} |\dot{\sigma}| = \sigma(1-\varepsilon) - \sigma(\varepsilon) \le \pi/2 + \frac{1+\alpha}{\alpha}.$$

Thus we can take $\varepsilon = 0$ and we deduce that

$$\int_0^1 |\dot{\beta}|\,dt = \int_0^1 ||\beta|^\cdot|\,dt \le \pi/2 + 2\frac{1+\alpha}{\alpha} \le C. \tag{20}$$

Step 2. The function

$$G(t) = \int_{\Omega(t)} |\nabla f|^2\,dx$$

is absolutely continuous. To see this let

$$U = \{(y_1, y_2) : -l^-(y_2) < y_1 < l^+(y_2),\ 0 < y_2 < 1\}$$

and consider the change of variables $\Phi : U \to \Omega(1)$ given by

$$\Phi\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} 0 \\ y_2 \end{pmatrix} + y_1\begin{pmatrix} 1 \\ s(y_2) \end{pmatrix}. \tag{21}$$

Thus the image of $y_1 \mapsto \Phi(y_1, y_2)$ is exactly the line segment through $(0, y_2)$ on which f is constant. Since these line segments do not intersect Φ is a bijection. Moreover

$$\nabla\Phi(y) = \begin{pmatrix} 1 & 0 \\ s(y_2) & 1 + y_1\dot{s}(y_2) \end{pmatrix}, \quad \det \nabla\Phi(y) = 1 + y_1\dot{s}(y_2).$$

In fact non-intersection of the line segments implies that $1 + y_1\dot{s}(y_2) > 0$ in U, so that Φ is locally Bilipschitz. Thus the area formula yields

$$G(t) = \int_0^t \int_{-l^-(y_2)}^{l^+(y_2)} |\nabla f|^2(\Phi(y_1, y_2))(1 + y_1\dot{s}(y_2))\, dy_1\, dy_2.$$

Since the integrand is nonnegative and $G(t) \le G(1) \le C$ Fubini's theorem shows that the inner integral defines a function in $L^1(0, 1)$. Thus $G \in W^{1,1}(0, 1)$ and

$$\dot{G}(t) = \int_{-l^-(t)}^{l^+(t)} |\nabla f|^2(\Phi(y_1, y_2))(1 + y_1\dot{s}(y_2))\, dy_1, \tag{22}$$

$$\int_0^1 |\dot{G}|\, dt = \int_0^1 \dot{G}\, dt = G(1) \le \int_\Omega |\nabla f|^2\, dx. \tag{23}$$

Step 3. Estimates of $\dot{F}$ in terms of $\dot{G}$.

By the definition (21) of Φ we have

$$F(y_2) = f(\Phi(y_1, y_2)) \quad \text{for } y \in U. \tag{24}$$

Since Φ is locally Bilipschitz the function $f \circ \Phi$ is in $W^{1,2}_{\text{loc}}$ and thus absolutely continuous on a.e. interior line segment in y_2 direction. Thus F is absolutely continuous on every interval $(\varepsilon, 1 - \varepsilon)$, with $\varepsilon > 0$, and by the chain rule

$$\dot{F}(y_2) = (\partial_2 f)(\Phi(y))(1 + y_1\dot{s}(y_2))$$

for a.e. y_2. Differentiating (24) with respect to y_1 we see that

$$0 = (\partial_1 f)(\Phi(y)) + s(y_2)(\partial_2 f)(\Phi(y)).$$

Thus

$$|\dot{F}|(y_2) \le (1 + s^2(y_2))^{-1/2}|\nabla f|(\Phi(y))(1 + y_1\dot{s}(y_2)). \tag{25}$$

Let

$$h(y_2) = \int_{-l^-(y_2)}^{l^+(y_2)} \frac{dy_1}{1 + y_1\dot{s}(y_2)}. \tag{26}$$

Now divide (25) by $(1 + y_1\dot{s}(y_2))$, integrate in y_1 and use the Cauchy-Schwarz inequality in connection with (22). This yields

$$|\dot{F}|(y_2)h(y_2) \le (1 + s^2(y_2))^{-1/2}|\dot{G}(y_2)|^{1/2}h^{-1/2}(y_2) \tag{27}$$

and hence (15). To verify (16) it suffices to restrict the integral in (26) to $(0, l^+(y_2))$ or to $(-l^-(y_2), 0)$.

To prove (14) we use the change of variables $s = \tan\beta$ and obtain

$$|\dot{F}| \le \omega^{1/2}|\dot{G}|^{1/2}, \tag{28}$$

where

$$\omega(t) = \frac{\dot{s}(t)}{1+s^2(t)} \frac{1}{\ln(1+\bar{l}(t)|\dot{s}(t)|)} = \frac{|\dot{\beta}(t)|}{\ln(1+(1+s^2(t))\bar{l}(t)|\dot{\beta}(t)|)} \quad .$$

If $s \geq 0$ we can now apply (18) to estimate l^+ from below; is $s < 0$ we have a similar bound for l^-. Thus in eihter case we get

$$(1+s^2(t))\bar{l}(t) \geq \min\left(\frac{1+s^2(t)}{s(t)}, t^{\frac{1}{1+\alpha}}\right) \geq t^{\frac{1}{1+\alpha}}.$$

Now using (28) and (23) we get

$$|F(t_0+\tau) - F(t_0)| \leq \left(\int_{t_0}^{t_0+\tau} \frac{|\dot{\beta}|}{\ln(1+t^{1/(1+\alpha)}|\dot{\beta}|)}\, dt\right)^{1/2} ||\nabla f||_{L^2(\Omega)}$$

and the assertion (14) follows from the L^1 bound (20) on $\dot{\beta}$ and Proposition 10 below.

Step 4. It remains to remove the additional assumption that through every point $(0, t)$ there exists a line segment (touching $\partial\Omega$ on both ends) on which f is constant. Let Ω_1 be the open set on which f is locally constant. Then the line segments considered above exist only for $(0, t) \in E = (\{0\} \times (0, 1)) \setminus \Omega_1$. Hence the slope function is only defined on E. On the maximal intervals (a, b) of $\Omega_1 \cap (\{0\} \times (0, 1))$ we define an interpolation as follows. If $s(a) = s(b)$ set $\tilde{s} = s(a) = s(b)$ in (a, b). If $s(a) \neq s(b)$ let $\bar{x} = (\bar{x}_1, \bar{x}_2)$ denote the intersection point of the lines through $(0, a)$ and $(0, b)$. Note that this intersection point must lie outside Ω by property (L). Define $\tilde{s}(t)$ such that the line through $(0, t)$ goes through $\bar{x}$, i.e.

$$\tilde{s}(t) = \frac{\bar{x}_2 - t}{\bar{x}_1}, \quad \text{for } t \in (a, b).$$

In particular $\tilde{s}$ is affine on (a, b) and satisfies the same estimates as the function s considered in Step 1.

Thus if we define Φ using the extension $\tilde{s}$ then Φ is again a bijection and locally Bilipschitz, and (22) and (24) hold. Thus F is absolutely continuous (on compact subintervals) and (25) holds with s replaced by $\tilde{s}$. Now we can conclude as before. □

Proposition 10. *Let* $\beta \in W^{1,1}(0, 1)$, $0 < \gamma < 1$. *If* $0 < t_0 < t_0 + \tau < 1$ *then, for all* $\eta \in (\gamma, 1)$,

$$\int_{t_0}^{t_0+\tau} \frac{|\dot{\beta}|}{\ln(1+t^\gamma|\dot{\beta}|)}\, dt \leq \frac{1}{(\eta-\gamma)} \frac{1}{\ln(1/\tau)} \int_0^1 |\dot{\beta}|\, dt + \frac{1}{1-\eta} \frac{1}{\ln 2} \tau^{1-\eta}. \tag{29}$$

Proof. After replacing $\beta(t)$ by $\beta(t_0+t)$ we may suppose $t_0 = 0$. We subdivide $(0, \tau)$ as follows

$$E_1 = \{t \in (0, \tau) : |\dot{\beta}| \leq t^{-\eta}\}, \quad E_2 = \{t \in (0, \tau) : |\dot{\beta}| > t^{-\eta}\}.$$

Since $x \mapsto \ln(1+x)$ is concave on $\mathbb{R}_+$ we have

$$\ln(1+x) \geq \frac{\ln(1+y)}{y} x, \quad \text{if } 0 < x \leq y.$$

Applying this with $x = t^\gamma|\dot{\beta}|$ and $y = t^{\gamma-\eta}$ we see that on E_1 the integrand is bounded by

$$\frac{|\dot{\beta}|}{t^\gamma|\dot{\beta}|} \frac{t^{\gamma-\eta}}{\ln(1+t^{\gamma-\eta})} \leq \frac{1}{\ln 2} t^{-\eta}.$$

On E_2 the integrand is trivially bounded by

$$\frac{|\dot{\beta}|}{\ln(1+t^{\gamma-\eta})} \leq \frac{|\dot{\beta}|}{(\eta-\gamma)\ln(1/\tau)}.$$

Thus (29) follows. □

We are now in a position to combine the local results to obtain the global estimate.

Proof of Theorem 6. First note that we may assume without loss of generality that S is such that at each boundary point p the set S contains the parabolic domain Ω (up to rigid motions) considered in Lemma 9. Indeed, by assumption there exists a radius r_0 and a Hölder constant M such that for each boundary point p there exists an orthornomal coordinate system and a $C^{1,\alpha}$ function g such that $p = (0,0)$, $\partial S \cap B(0, r_0) \subset \operatorname{graph} g$, $S \cap B(0, r_0)$ lies above graph g, $g'(0) = 0$ and $|g'(x_1) - g'(y_1)| \leq M|x_1 - y_1|^\alpha$. Now let $R = 4^{1+\alpha}\max(r_0, M^{1/\alpha})$ and consider the rescaled set RS. Then at each boundary point of RS the set RS contains the desired standard parabola Ω (up to a rigid motion). Hence we can work with RS and rescale at the end (note that this only effects the radius $R_0(S)$ in the statement since the constant C in the oscillation estimate is invariant under dilations). Note also that using the same rescaling we may in addition assume that the Hölder seminorm of g' is bounded by 1.

Now consider points $p, q \in S$. We may assume that $\operatorname{dist}(p, \partial S) \geq \operatorname{dist}(q, \partial S)$ and we set

$$d = \operatorname{dist}(p, \partial S) \geq \operatorname{dist}(q, \partial S), \quad r = |p - q|.$$

We claim that for $r \leq \bar{r}$ (where $\bar{r}$ is a constant only depending on S) we have

$$|f(p) - f(q)| \leq C \ln^{-1/2}(1/r). \tag{30}$$

Case 1. Suppose $r \leq d/4$. If $d \geq 1$ we can use the interior estimate (12) since $q \in \overline{B(p,r)}$ and $B(p,d) \subset S$. If $d < 1$ let $\bar{p}$ be a boundary point which has minimal distance to p and consider a coordinate system centered at $\bar{p}$. Thus p has coordinates $(0, d)$. Let $p' = (0, d - 2r)$. Then $|p' - q| \leq 2r + |p - q| \leq 3r$. Thus $p, q \in B^+(p', 3r)$. On the other hand the parabola Ω and hence the set S contains the half-disc $B^+(p', R)$ with $R = (d - 2r)^{1/(1+\alpha)} \geq r^{1/(1+\alpha)}$. Thus (30) follows from Proposition 8.

Case 2. Suppose that $d = 0$, i.e. $p, q \in \partial S$.

Let ν_p and ν_q denote the inner normals at p and q, respectively, and set

$$p' = p + 16r\nu_p, \quad q' = q + 16r\nu_q.$$

Since $|\nu_p - \nu_q| \leq r^\alpha$ we may suppose that $|p' - q'| \leq 2r$. Moreover $\mathrm{dist}(p', \partial S) \geq 8r$ and $\mathrm{dist}(q', \partial S) \geq 8r$ (for sufficiently small r). Hence by Case 1

$$|f(p') - f(q')| \leq C \ln^{-1/2}(1/r).$$

On the other hand Lemma 9 shows that $|f(p') - f(p)| \leq C \ln^{-1/2}(1/16r)$ and the same estimate holds for $f(q) - f(q')$. Thus (30) follows.

Case 3. Suppose $d < 4r$.

Let $\bar{p}, \bar{q}$ points on ∂S which have minimal distance from p and q, respectively. Then $|q - \bar{q}| \leq |p - \bar{p}| \leq d$. Thus $|\bar{p} - \bar{q}| \leq r + 2d \leq 9r$. Now $f(\bar{p}) - f(\bar{q})$ can be estimated as in Case 2, while $f(p) - f(\bar{p})$ and $f(q) - f(\bar{q})$ are estimated by Lemma 9. This finishes the proof of (30) and hence of Theorem 6. □

4. Approximation by Lipschitz functions

In this section we prove the following approximation result, stated already in the introduction as Theorem 2.

Theorem 11. *Suppose that $S \subset \mathbb{R}^2$ is a bounded Lipschitz domain and $V \in W^{2,2}(S)$ satisfies*

$$\det \nabla^2 V = 0.$$

Then there exists an increasing sequence of open subset S_k and maps $V_k \in W^{2,2}(S)$ such that

$$||\nabla V_k||_{L^\infty(S)} \leq k, \quad V_k = V \quad \textit{in } S_k, \tag{31}$$

$$\nabla^2 V_k = 0 \quad \textit{a.e. on } S \setminus S_k, \tag{32}$$

$$\bigcup_{k=1}^{\infty} S_k = S. \tag{33}$$

In particular we have $\det \nabla^2 V_k = 0$, $||\nabla V_k||_{L^2} \leq ||\nabla V||_{L^2}$ *and* $V_k \to V$ *in* $W^{2,2}(S)$.

Remark 12. If $\Gamma \subset \partial S$ is a finite union of intervals and the trace of ∇V on ∂S satisfies $||\nabla V||_{L^\infty(\Gamma)} \leq M$ then $V_k = V$ and $\nabla V_k = \nabla V$ in an open subset of S (with Lipschitz boundary) whose boundary contains Γ for sufficiently large k. In particular the equality holds in the sense of trace on Γ.

Proof. We will use the assumption $\det \nabla^2 V = 0$ only to conclude that $V \in C^1(S)$ (see Proposition 5) and that at each point $x \in S$ either ∇V is locally constant or ∇V is constant on a line segment through x which intersects ∂S at both ends (Condition (L)). Note that both these properties remain true if we subtract an affine function from V.

Let U be an open ball whose closure is contained in S. After subtracting an affine map from V we may suppose that

$$\int_U \nabla V \, dx = 0.$$

Together with the interior estimate (12) this shows that

$$|\nabla V| \leq C \quad \text{in } U.$$

Now consider the set $U_k = \{x \in S : |\nabla V(x)| < k\}$. This set is open since $V \in C^1(S)$ by Proposition 5 and for large enough k it contains U. Let S_k denote the connected component of U_k which contains U.

Step 1. We claim that

(i) $\partial S_k \cap S$ is a union of line segments on which ∇V is constant and satisfies $|\nabla V| = k$. Moreover each segment intersects ∂S at both of its endpoints;

(ii) $\cup_{k=1}^{\infty} S_k = S$.

To verify this consider $\bar{x} \in \partial S_k \cap S$. By the continuity of ∇V we have $|\nabla V|(\bar{x}) = k$. Thus ∇V cannot be locally constant near $\bar{x}$ (otherwise $\bar{x} \notin \partial S_k$). Hence by Theorem 4 there exists a line segment l which intersects ∂S at both endpoints and along which ∇V is constant. In particular $|\nabla V| = k$ on l so that $l \cap S_k = \emptyset$. We claim that $l \subset \partial S_k$. To see this note that there exists a sequence of points $x_j \in S_k$ with $x_j \to \bar{x}$ such that ∇V is constant on a line segment l_j through x_j (which extends up to ∂S). We can take, for examle, x_j as a point in $\{x \in S_k : |\nabla V(x)| \leq k - 1/j\}$ which has minimal distance from $\bar{x}$. In view of this minimality property ∇V cannot be constant near x_j and hence the desired line segment l_j exists. The segments l_j cannot intersect l (in S) and their lengths are bounded from below. Thus they must converge to l (e.g. in the Hausdorff sense) since $x_j \to \bar{x}$. Since $l_j \subset S_k$ and $l \cap S_k = \emptyset$ we conclude that $l \subset \partial S_k$.

It remains to show that the S_k exhaust S. Since $V \in C^1(S)$ we have $|\nabla V| = k$ on $\partial S_k \cap S$ and thus

$$d_k := \sup_{x \in \partial S_k} \operatorname{dist}(x, \partial S) \to 0. \tag{34}$$

Thus ∂S_k does not intersect the set $\{x \in S : \operatorname{dist}(x, \partial S) > 2d_k\}$. Hence either S_k contains this set or it does not intersect it. Since S_k contains U the latter is impossible for sufficiently large k. This shows that $\cup_{k=1}^{\infty} S_k = S$.

Step 2. Next we show that (for sufficiently large k) the set $S \setminus \overline{S_k}$ is a union of pairwise disjoint open sets W_j and $\partial W_j \cap S$ is exactly one of the line segments in $\partial S_k \cap S$ (see Fig. 2). To see this consider one such segment l. By (34) each point in l has at most distance d_k from ∂S. Hence the endpoints p and q of l lie on the same component of ∂S (here and in the following we always assume that k is sufficiently large). We claim that there exists an open and connected set W whose boundary consists of l and a curve $\gamma \subset \partial S$ from p to q and which satisfies

$$\operatorname{dist}(x, \partial S) \leq C d_k, \quad \forall x \in W, \tag{35}$$

$$l \subset \operatorname{int}(S_k \cup \overline{W}). \tag{36}$$

To see this note that by definition of a set with Lipschitz boundary a neighbourhood of ∂S can be covered by open balls B_i such that $\partial S \cap B_i$ is contained in a Lipschitz graph $\{(x_1, g(x_1)\}$ (in a suitable orthonormal coordinate system) with $\operatorname{Lip} g \leq L$ and that $S \cap B_i$ lies above that graph. If l is sufficiently short then both l and the boundary arc connecting p and q lie in a single such chart and we can take

$$W = \{x_1 \in [a, b] : g(x_1) < x_2 < h(x_1)\},$$

where $p = (a, g(a)), q = (b, g(b))$ and where the affine function h represents the line segment l.

If l is not contained in a single chart we can subdivide l into a disjoint union of segments l_J, $J = 1, \ldots, m$ which do lie in a single chart. Let p_J and $q_J = p_{J+1}$ be the endpoints of l_J and let $\overline{p_J}$ and $\overline{q_J}$ be points on ∂S which are closest to them (these may not be unique but any choice will do), see Figure 2. Let $\overline{W_J}$ be the closed deformed rectangle bounded by l_J, the path γ_J (in the chart considered) from $\overline{p_J}$ to $\overline{q_J}$ and the line segments $[p_J, \overline{p_J}]$ and $[q_J, \overline{q_J}]$ (for the first segments l_1 and the last segment l_k the rectangle degenerates into a triangle). Now $W = \operatorname{int}(\cup W_J)$ has then desired properties (35) and (36).

Next we claim that

$$S_k \cap W = \emptyset. \tag{37}$$

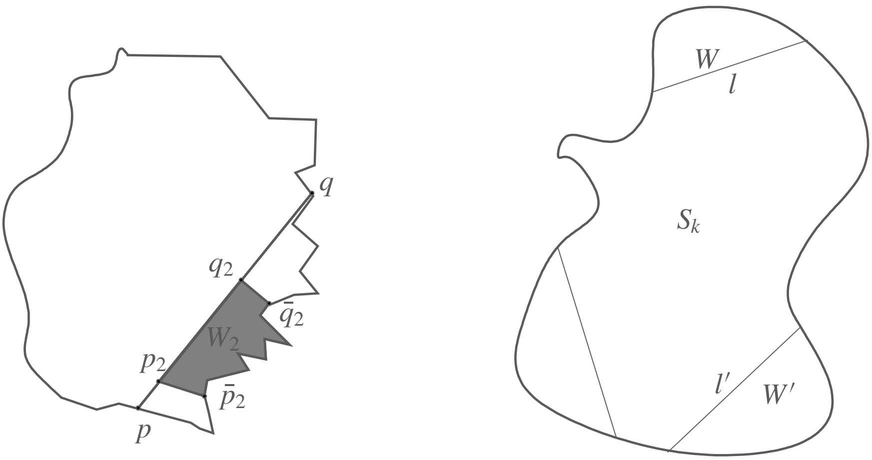

Fig. 2 **a** Construction of the set W for a long line segment l **b** Sets W and W' for different boundary segments l and l'.

Note that $S_k \cap \partial W = \emptyset$. Since S_k is connected failure of (37) would imply $S_k \subset W$. But this is impossible since S_k contains U while W satisfies (35). Consider now the line segments l and l' in ∂S_k and the corresponding sets W and W'. We claim that

$$W \cap W' = \emptyset, \quad \text{if } l \neq l'. \tag{38}$$

Suppose that $l \neq l'$. Then the segment l (without the endpoints) does not intersect $\partial W'$. Thus either $l \cap W' = \emptyset$ or $l \subset W'$. The latter possibility cannot occur since $S_k \cap W' = \emptyset$ by (37). Hence $W' \cap \partial W = W' \cap l = \emptyset$. As W' is connected this shows that either $W' \cap W = \emptyset$ or $W' \subset W$. If the former possibility does not occur then we can exchange the roles of W and W' and we get $W = W'$. Hence $l = l'$ and this contradiction proves (38).

Since each of the sets W has positive area it follows from (38) that ∂S_k consists of at most countably many line segments l_j. We finally claim that

$$S \subset S_k \cup \bigcup_j \overline{W_j}. \tag{39}$$

Denote the set on the right hand side by S'. Then $\partial S' \cap S \subset (\cup_j \partial W_j \cup \partial S_k) \cap S \subset \partial S_k$. It now follows from (36) that $\partial S' \cap S = \emptyset$. Hence $S \subset S'$ as claimed.

Step 3. Now we can easily define the approximations V_k. Let $f = \nabla V$. Since f is constant on the line segment l_j there exists a unique affine function $L_j : \mathbb{R}^2 \to \mathbb{R}$ such that $\nabla L_j = f$ and $L_j = V$ on l_j. Now set

$$V_k = \begin{cases} V & \text{on } \overline{S_k} \\ L_j & \text{on } \overline{W_j}. \end{cases} \tag{40}$$

It follows from (36) and (39) that V_k is well-defined and belongs to $W^{2,2}(S)$. Note that if two segments l and l' share a boundary point then the fact that $V \in W^{2,2}$ and the Poincaré inequality show that $f_{|l} = f_{|l'}$ and hence $L = L'$. Indeed in this case S contains a sector $T = \{x : x_0 + (r\cos\varphi, r\sin\varphi) : 0 < r < r_0, \varphi_0 < \varphi < \varphi_1\}$ which is bounded by (parts of) the segments l and l' and a circular arc. Let $a = f_{|l}$ and $a' = f_{|l'}$ and assume without loss of generality $x_0 = 0$. Application of the Poincaré inequality to the arc $\gamma_r = \{(r\cos\varphi, r\sin\varphi) : \varphi_0 < \varphi < \varphi_1\}$ yields

$$|a - a'|^2 \leq Cr \int_{\gamma_r} |\nabla^2 V|^2 \, d\mathcal{H}^1.$$

Dividing by r and integrating from 0 to r_0 we deduce that $a = a'$ since $\nabla^2 V \in L^2$. Similarly we see that L and L' must agree in the points where the segments l and l' touch.

We also see directly from (40) that $\nabla^2 V_k = \nabla^2 V$ in S_k and $\nabla^2 V_k = 0$ a.e. in $S \setminus S_k$ (since ∂S_k is a countable union of line segments and thus a two-dimensional null set). Moreover $|\nabla V_k| \leq k$. This finishes the proof of Theorem 11. $\square$

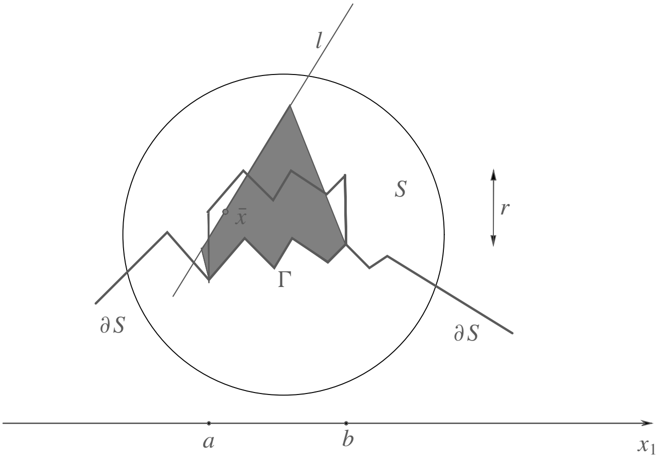

Fig. 3 Approximation near the boundary. The Poincaré inequality is applied in the shaded region

Proof of Remark 12. It suffices to consider the case that Γ is a single interval contained in a single Lipschitz chart of the boundary, i.e. $\partial S \cap B_R \subset \operatorname{graph} g$, $\Gamma = \operatorname{graph} g_{|[a,b]}$, $\operatorname{Lip} g \le L$, see Figure 3. Set

$$\Gamma_r = \{(x_1, x_2) : x_1 \in [a, b],\ g(x_1) < x_2 < g(x_1) + r\}.$$

We claim that for sufficiently small $r > 0$ we have

$$|\nabla V| < M + 1 \quad \text{in } \Gamma_r. \tag{41}$$

Once this is shown we conclude easily as follows. The sets S_k are increasing to S and therefore $S_k \cap \Gamma_r \neq \emptyset$ for all sufficiently large k. By (41) we have $\Gamma_r \cap \partial S_k = \emptyset$. Since Γ_r is connected this implies that $\Gamma_r \subset S_k$ and thus $V_k = V$ in Γ_r. Hence we have $V_k = V$ and $\nabla V_k = \nabla V$ on Γ in the sense of trace.

We prove (41) by contradiction. Let $\hat{x} \in \Gamma_r$ with $|\nabla V(\hat{x})| \ge M + 1$. We claim first that

$$\exists \bar{x} \in \Gamma_r \text{ such that } \nabla V(\bar{x}) = \nabla V(\hat{x}) \text{ and } \nabla V \text{ is constant on a line segment } l \\ \text{through } \bar{x} \text{ which intersects } \partial S \text{ at both endpoints.} \tag{42}$$

Once we have found such an $\bar{x}$ it follows from the Poincaré inequality in direction x_2 that the intersection of l with ∂S occurs outside Γ. Indeed suppose otherwise. We write $\bar{x} = (\bar{a}, \bar{z})$ with $\bar{a} \in (a, b)$. Suppose that the line l through $\bar{x}$ intersects Γ in the point $p = (\tilde{a}, g(\tilde{a}))$. We may assume that $p = (0, 0)$. Let $G = |\nabla V|$. Then $G(x_1, \frac{\bar{z}}{\bar{a}} x_1) = G(\bar{x}) \ge M + 1$, while $G(x_1, g(x_1)) \le M$, by assumption. Thus the Poincaré inequality in x_2 direction yields

$$1 \le \left| \frac{\bar{z}}{\bar{a}} x_1 - g(x_1) \right| \int |\nabla^2 V|^2(x_1, x_2)\, dx_2 \le \left(\left| \frac{\bar{z}}{\bar{a}} \right| + L \right) x_1 \int |\nabla^2 V|^2(x_1, x_2)\, dx_2$$

and dividing by x_1 and integrating over x_1 from 0 to $\bar{a}$ we reach a contradiction.

We now proceed to prove the claim. If ∇V is not locally constant near $\hat{x}$ then (42) follows directly from Theorem 4 with $\bar{x} = \hat{x}$. If ∇V is locally constant near $\hat{x}$ consider the open set $S_1 = \{x \in S : \nabla V \text{ is constant near } x\}$ and let U denote the connected component of S_1 which contains $\hat{x}$. If $\partial U \cap \Gamma_r \neq \emptyset$ we can take $\bar{x} \in \partial U \cap \Gamma_r$ and apply again Theorem 4. If $\partial U \cap \Gamma_r = \emptyset$ then the connectedness of Γ_r shows that $\Gamma_r \subset U$. Thus $|\nabla V| = |\nabla V(\hat{x})| \geq M + 1$ in Γ_r. This contradicts the assumption that ∇V satisfies $|\nabla V| \leq M$ on Γ (in the sense of trace). Thus (42) holds.

Now we obtain (41) easily by an application of Poincaré's inequality. Indeed if the line l has slope between $-2L$ and $2L$ the application of Poincaré's inequality (in the x_2-direction) in the region between Γ and l yields, as in the calculation above,

$$||\nabla^2 V||^2_{L^2} \geq \int_0^{b-a} \frac{dx_1}{r + 3Lx_1} = \frac{1}{3L} \ln\left(1 + \frac{3L(b-a)}{r}\right).$$

Thus we obtain a contradiction if r is chosen sufficiently small. If l has slope larger than $2L$ (this can only happen if $\bar{x}$ is close to the left endpoint $(a, g(a))$ of Γ) then we can apply the Poincaré inequality along a family of lines with slope $-2L$ which connect Γ and l (equivalently we could slightly tilt the picture in Figure 3 so that the slope of l is $2L$ in the tilted picture and apply again the Poincaré inequality in x_2 direction). If the slope of l is less than $-2L$ then we apply the Poincaré inequality on a family of lines with slope $2L$. This finishes the proof of (41) and thus yields the assertion. □

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