Annals of Global Analysis and Geometry **23:** 1–12, 2003.

Weak Density of Smooth Maps in $W^{1,1}(M, N)$ for Non-Abelian $\pi_1(N)$

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(Received: 20 July 2001; accepted: 19 April 2002)

Communicated by: T. River (Cachau)

Abstract. We prove that smooth maps are dense in the sense of biting convergence in $W^{1,1}(M, N)$ when M and N are compact Riemannian manifolds and N is closed.

Mathematics Subject Classifications (2000): 46-XX, 49-XX.

Key words: topological singularities, minimal connections, density of smooth maps, Sobolev spaces between manifolds.

1. Introduction

Let M and N smooth compact Riemannian manifolds such that N is closed and isometricly embedded in $\mathbb{R}^N$. Set

$$W^{1,1}(M, N) := \{u \in W^{1,1}(M, \mathbb{R}^N); u(x) \in N \text{ for a.e. } x \in M\}.$$

This space inherits the strong and the weak topology of $W^{1,1}(M, \mathbb{R}^N)$ and is closed under the weak convergence of maps in $W^{1,1}$. The energy of a map $u \in W^{1,1}(M, N)$ is defined to be $\int_M |\nabla u|$.

Based on the work of Schoen and Uhlenbeck [20], Zheng and Bethuel [6] and Bethuel [2], we know that smooth maps from $\mathbf{B}^n$ into N, where $\mathbf{B}^n$ is the unit disk in $\mathbb{R}^n$, are not dense in $W^{1,1}(\mathbf{B}^n, N)$ if $\pi_1(N) \neq 0$. In fact, they showed that the lack of approximability is due to local realizations by $u \in W^{1,1}(\mathbf{B}^n, N)$ of nonzero elements of $\pi_1(N)$ around points in $\mathbf{B}^n$. In particular, they proved that if $\pi_1(N) = 0$ then any map in $W^{1,1}(\mathbf{B}^n, N)$ can be approximated by smooth maps for the strong topology. A major question would be to determine a criteria for a map to be approximable by smooth maps in $W^{1,1}(\mathbf{B}^n, N)$, i.e. we try to define $\mathbf{S}_u$, 'the topological singular set' of u, which would be equal to zero if and only if u is a strong limit of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$.

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In the case $\pi_1(N) \neq 0$, one can approximate the maps in $W^{1,1}(\mathbf{B}^n, N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth $(n-2)$-dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^\infty(\mathbf{B}^n, N)$. A map $v \in R^\infty(\mathbf{B}^n, N)$ realizes elements σ_x of $\pi_1(N, y)$ on the circles centered at any point $x \in \Sigma(v)$ and contained in the normal bidimensional plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is nontrivial, then v cannot be approximated by smooth maps in the strong topology (see [2]). In [18], we observed that if $\pi_1(N)$ is Abelian, one can assign to v a $\pi_1(N)$-chain which is carried by $\Sigma(v)$ with 'multiplicity' σ_x at each point x of $\Sigma(v)$. This $\pi_1(N)$-chain is called the topological singular set $\mathbf{S}_v$ of v in $R^\infty(\mathbf{B}^n, N)$. Moreover, for a sequence of maps $v_m \in R^\infty(\mathbf{B}^n, N)$ converging strongly to any $u \in W^{1,1}(\mathbf{B}^n, N)$, $\mathbf{S}_{v_m}$ converges in the flat norm to a unique flat $\pi_1(N)$-chain $\mathbf{S}_u$ we called the topological singular set of u.

This approach confronts important obstacles when $\pi_1(N)$ is not Abelian. The major problem is the following: If $\pi_1(N)$ is Abelian, its elements are well defined independent of the choice of the base point in N, i.e. we can define isomorphisms $\gamma_\#$ between $\pi_1(N, y)$ and $\pi_1(N, y')$ with the aide of smooth curves γ joining y and y' in N. These isomorphisms do not depend of the choice of γ and so we can identify $\pi_1(N, y)$ and $\pi_1(N, y')$ in a natural manner. In this way, e.g. we can compare the topological singularity of $u \in R^\infty(\mathbf{B}^2, \mathbb{RP}^2)$ around different points in the square $\mathbf{B}^2$ without ambiguity, though the values of u in $\mathbb{RP}^2$ near these points might differ. But, if $\pi_1(N)$ is not Abelian, there is no canonical isomorphism between $\pi_1(N, y)$ and $\pi_1(N, y')$ for two different points $y, y' \in N$. The isomorphisms $\gamma_\#$ would depend on the homotopy class of γ and even a closed curve γ joining y to itself may produce a nontrivial isomorphism of $\pi_1(N, y)$ onto itself. So, talking about the topological type of a singularity without fixing the base points in $\mathbf{B}^n$ and in N is impossible and we can neither compare the topological type of different singularities nor talk about connecting them by chains with coefficients in $\pi_1(N)$ as before.

Another problem we encounter in the study of this case is that $u \in R^\infty(\mathbf{B}^n, N)$ may have singularities of the type $aba^{-1}b^{-1}$ which are not removable by strong convergence of smooth maps. Meanwhile, following the method used in [18], the conjugation of u with p^a (or p^b), the projections of N on the generating cycles of a (or b), will not 'see' these singularities in the first instance, since $p^a \circ u$ (or $p^b \circ u$) would realize the cycles aa^{-1} (or bb^{-1}) in their respectable circle-type targets.

In this way, the question of defining a topological singular set for maps in $W^{1,1}(\mathbf{B}^n, N)$ is still open for non-Abelian $\pi_1(N)$. In this paper, we try to pave the way for understanding the situation by answering another related question. If $\pi_1(N)$ is Abelian, we can prove that for any map $u \in W^{1,1}(\mathbf{B}^n, N)$, there is a sequence of smooth maps, $v_m \in C^\infty(\mathbf{B}^n, N)$, such that u is the $W^{1,1}$-weak limit of v_m outside arbitrary small positive measure subsets of $\mathbf{B}^n$ (see Definition 1.1 below). The method consists in controlling the mass of chains which connect the singular chain of a map $u \in R^\infty(\mathbf{B}^n, N)$ to the boundary of $\mathbf{B}^n$ and then removing the singularities, spending an energy proportional to the mass of these connections (see [18]). The question is then whether this method can be modified to prove the same result for the non-Abelian $\pi_1(N)$ case.

For surmounting the above described problems for non-Abelian $\pi_1(N)$, we should introduce new elements into the proof. In fact, we search a kind of connecting set $A_u \subset \mathbf{B}^n$ of dimension $n-1$ for the singularities of a map $u \in R^\infty(\mathbf{B}^n, N)$ so that for any point $x \in A_u$ we can identify $a(x)$: the elements of $\pi_1(N, u(x))$ which should be introduced into u (transversally to A_u at x) such that the singularities of u are removed. These connecting sets should also take into account the problems provoked by $aba^{-1}b^{-1}$-type singularities described above. And, last but not least, the one-energy of inserted curves producing $a(x)$ at $x \in A_u$ should be controlled uniformly (independent of the choice of x and u) so that the total energy of the modification be uniformly proportional to the volume of A_u, which in its turn is controlled by the energy of u. All this is possible for a converging sequence $u_m \to u \in W^{1,1}(\mathbf{B}^n, N)$.

At last, for obtaining the same results for any smooth compact manifold M as the domain, one should be careful as there may be some global topological obstructions we did not consider up to now. We refer to the recent work of Hang and Lin [16, 17] where they show that the absence of 'local' topological obstructions does not mean the approximability by smooth maps in the strong topology. As we shall see below, the method exposed in this paper allow us to remove these kinds of singularities too. So here is the main results of this paper:

DEFINITION 1.1. Let Ω be a domain in $\mathbb{R}^n$ and let u_m be a bounded sequence in Ł$^1(\Omega)$. u_m is said to converge in the biting sense to $u \in L^1(\Omega)$ if for every $\varepsilon > 0$ there exists a measurable set $E \subset \Omega$ such that $\mu(E) < \varepsilon$ and $u_m \rightharpoonup u$ weakly in $L^1(\Omega \backslash E)$.

THEOREM I. *Let M and N be smooth Riemannian manifolds for which $\partial N = \emptyset$. Then for every $u \in W^{1,1}(M, N)$ there is a sequence of maps $u_m \in C^\infty(M, N)$ such that ∇u_m tend to ∇u in the biting sense.*

Remark 1.1. Hang has recently proved that sequential weak closure of smooth maps in $W^{1,1}(M, N)$ coincides with the strong closure of smooth maps [15], thus when $\pi_1(N) \neq 0$ the density of smooth maps in the biting sense may be the best one could hope for $W^{1,1}(M, N)$.

Assume that ∂M is not empty. We may also ask the same questions about the spaces of maps with fixed boundary value: For $\varphi \in C^\infty(\partial M, N)$, admitting a smooth extension ϕ: $M \to N$, we define

$$C^\infty_\varphi(M, N) := \left\{u \in C^\infty(M, N); \quad u = \varphi \text{ on } \partial M\right\}$$

and

$$W^{1,1}_\varphi(M, N) := \left\{u \in W^{1,1}(M, N); \quad u = \varphi \text{ a.e. on } \partial M\right\}.$$

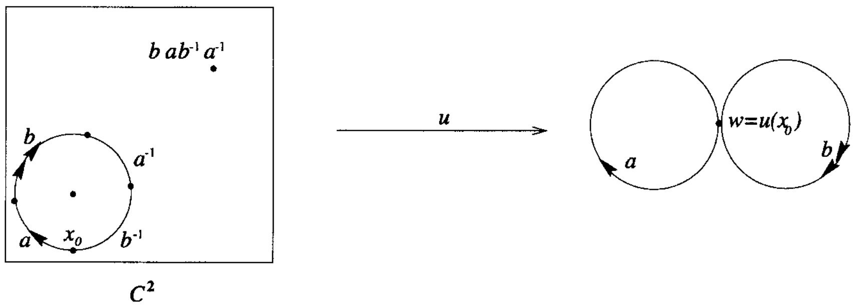

Figure 1. An $(aba^{-1}b^{-1})$-type singularity dipole.

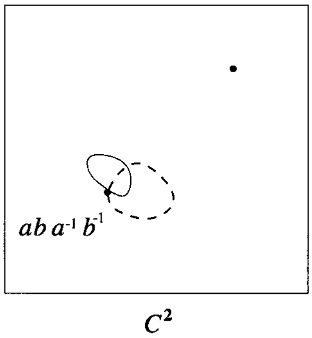

Figure 2. A bad connecting set for the dipole (not suitable for removing the singularities).

THEOREM Ibis. *Let M and N be smooth Riemannian manifolds for which $\partial N = \emptyset$. Assume that $\varphi \in C^\infty(\partial M, N)$ is smoothly extendable onto M. Then for every $u \in W^{1,1}_\varphi(M, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(M, N)$ such that ∇u_m tend to ∇u in the biting sense.*

As a simplified example, consider the space $W^{1,1}(\mathbf{B}^n, \mathcal{S}_2)$, where $\mathcal{S}_2 := S^1_a \vee S^1_b$ is the bouquet of two circles based on the point $w \in \mathbb{R}^2$. $\pi_1(\mathcal{S}_2, w)$ is the free (thus non-Abelian) group generated by two generators a and b. Let p^a and p^b be the projection of $\mathcal{S}_2$ onto S^1_a and S^1_b. The idea is to associate to any sequence

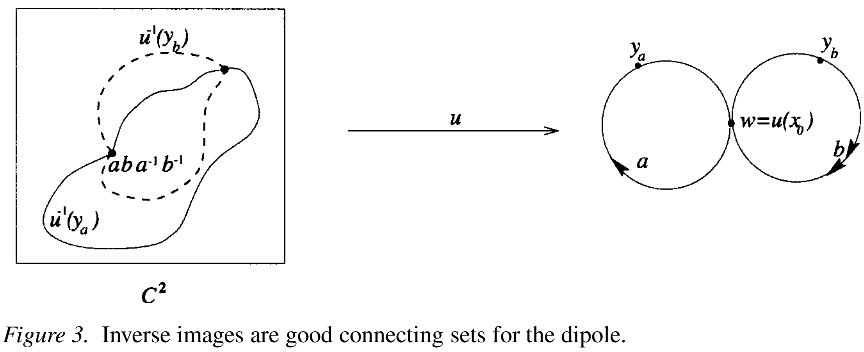

Figure 3. Inverse images are good connecting sets for the dipole.

$u_m \in R^{\infty}(\mathbf{B}^n, \mathcal{S}_2)$, converging strongly to $u \in W^{1,1}(\mathbf{B}^n, \mathcal{S}_2)$, two points $y_a \in S_a^1$ and $y_b \in S_b^1$ such that

$$A_{u_m} := A_{u_m}^a \cup A_{u_m}^b := (p^a \circ u_m)^{-1}(y_a) \cup (p^b \circ u_m)^{-1}(y_b)$$

is a finite union of smooth submanifolds of $\mathbf{B}^n$ and that for a uniform constant $C > 0$

$$\mathrm{vol}(A_{u_m}) \leq C \int_{\mathbf{B}^n} |\nabla u| + C.$$

Then the topological considerations detailed in the paper show that A_{u_m} satisfy the above necessary conditions for suitable connecting sets. Observe that as the image of these 'connections' are constant in $\mathcal{S}_2$, the homotopy groups $\pi_1(\mathcal{S}_2, u_m(x))$ for $x \in A_{u_m}$ would have a fixed base point. For a visualization of this problem compare Figures 1, 2 and 3.

Finally, we mention that the same questions about the density of smooth maps and the topological singularities can be asked about the functional spaces $H^{1/2}(M, N)$, which is also an interesting case. In the Abelian case, see the corresponding studies of Rivière in [19].

2. Preliminaries

2.1. THE NON-ABELIAN FUNDAMENTAL GROUP

Let N be a closed smooth manifold and $y, y' \in N$ two base points. Any curve

$$\gamma\colon [0, 1] \to N$$

for which $\gamma(0) = y$ and $\gamma(1) = y'$, induces a natural isomorphism

$$\gamma_{\#}\colon \pi_1(N, y') \to \pi_1(N, y),$$

which depends only on the homotopy class of γ. If $\pi_1(N, y)$ is Abelian, these isomorphisms are canonical, that is they do not depend on the choice of the curve γ. In this case we can talk about $\pi_1(N)$ without ambiguity. Otherwise, for referring to a specific element of $\pi_1(N)$, we are obliged to fix a base point for $\pi_1(N)$. Now let us assume that $y = y'$ and consider a curve γ as above. We have

$$\gamma_{\#}(a) = [\gamma]a[\gamma]^{-1}, \quad \forall a \in \pi_1(N, y), \tag{2.1}$$

where $[\gamma]$ is the homotopy class of γ in $\pi_1(N, y)$. Naturally if $\pi_1(N, y)$ is not Abelian, these isomorphisms may not be trivial for $[\gamma] \neq 0$. See [7, section VII.7] for more details.

2.2. THE SUBSPACE $r^{\infty}(m, n)$

DEFINITION 2.1. We say that $u \in W^{1,1}(M, N)$ is in $R^{\infty}(M, N)$ if u is smooth except on $B = \bigcup_{i=1}^{m} \sigma_i \cup B_0$, a compact subset of M, where $\mathcal{H}^{n-2}(B_0) = 0$ and the σ_i, $i = 1, \ldots, m$ are smooth embeddings of the unit disk of dimension $n - 2$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.

THEOREM II (Bethuel [2]). *$R^{\infty}(M, N)$ is dense in $W^{1,1}(M, N)$ for the strong topology.*

DEFINITION 2.2. Let $u \in R^{\infty}(M, S^1)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-2)$-vectorfield σ_i. For $a \in \sigma_i$ let N_a be any two-dimensional smooth submanifold of M, orthogonal to σ_i at a. Consider the embedded disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the 2-vectorfield $\mathbf{M}_a$ such that $(-1)^{n-1}\sigma_i(a) \wedge \mathbf{M}_a$ is the fixed orientation of M. Then the topological degree of u on the closed curve $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a and N_a for δ small enough. We call this integer the degree of u on σ_i and denote it by $\deg_{\sigma_i} u$.

THEOREM III (Almgren et al. [1]). *Let M be as above, $u \in R^{\infty}(M, S^1)$, then for any regular value $y \in S^1$,*

$$\partial[[u^{-1}(y)]] - [[u^{-1}(y)]] \llcorner \partial M = \sum_{i=1}^{m} (\deg_{\sigma_i} u)\, [[\sigma_i]]$$

and

$$\int_{S^1} \mathcal{H}^{n-1}(u^{-1}(y))\, \mathrm{d}y \leq \int_M |\nabla u|\, \mathrm{d}\,\mathrm{vol}_M.$$

3. Proof of Theorem I

As in the case where $\pi_1(N)$ is Abelian, we should prove the existence of sets with bounded volume, connecting the singularities of a map in $R^{\infty}(M, N)$, along which we can modify the map for removing its singularities. Meanwhile, for some technical reasons, we should use the same process for the elements of any strongly convergent sequence $u_m \in R^{\infty}(M, N)$ when defining these sets.

Let us consider any map $u \in W^{1,1}(M, N)$ and a sequence of maps $u_m \in R^{\infty}(M, N)$ converging strongly to u. As we mentioned above, such a sequence always exist. We should show the existence of smooth maps $v_m \colon M \to N$, such that ∇v_m tend in the biting sense to ∇u.

Step 1: Projection of maps into some one skeleton of N
Consider some triangulation of N and for $1 \le l \le k$, let N^l be the l-skeleton of N. So $N = N^k$. Observe that by [21, theorem (1.6), p. 215], the homomorphism

$$\chi\colon \pi_1(N^1, y) \to \pi_1(N, y), \tag{3.1}$$

induced by the injection map $i\colon N^1 \to N$, is onto. Also using [12, corollary 3.5, p. 38], N^1 is of the homotopy type of a bouquet of circles and we obtain that $\pi_1(N^1)$ is finitely generated. Let $f\colon N^1 \to \mathcal{S}_\beta := \bigvee_{i=1}^{\beta} S_i^1$ be a homotopy equivalence between N^1 and the bouquet of β circles, $S_1^1, \ldots, S_\beta^1$, embedded in some Euclidean space and based on the fixed point w.

Let $\mathbf{B}^l$ be the unit disk in $\mathbb{R}^l$. We denote

$$U^l := \left\{ (x, y) \in \mathbf{B}^l \times \mathbf{B}^l \, ; \; x \neq y \right\}.$$

DEFINITION 3.1. For $(x, y) \in U^l$, we define $p(x, y)$ to be the unique point on $\partial\mathbf{B}^l$ which is on the ray from x to y.

Let us write

$$N^l = \bigcup_{i=1}^{s_l} \xi_i^l(\mathbf{B}^l),$$

where

$$\xi_i^l\colon \mathbf{B}^l \to N_i^l := \xi_i^l(\mathbf{B}^l), \quad i = 1, \ldots, s_l$$

are diffeomorphisms and each two N_i^l are rather disjoint or intersecting on a lower dimensional face in N^{l-1}. Let $w \in N_1^l \times \cdots \times N_{s_l}^l$, $w = (w_1, \ldots, w_{s_l})$ be such that $w_i \notin N^{l-1}$. Define

$$p_w^l\colon N^l \backslash \{w_1, \ldots, w_{s_l}\} \to N^{l-1}$$

as follows:

$$p_w^l(y) := \begin{cases} \xi_i^l(p((\xi_i^l)^{-1}(w_i), (\xi_i^l)^{-1}(y))), & \text{if } y \in N_i^l \backslash N^{l-1}, \\ y, & \text{otherwise,} \end{cases}$$

where p is the projection defined in Definition 3.1. We have this useful result, already proved in [18]: For any map $u \in W^{1,1}(M, N^l)$

$$\int_{N_{1,\varepsilon}^l \times \cdots \times N_{s_l,\varepsilon}^l} \int_M |\nabla(p_w^l \circ u)| \, \mathrm{d}\,\mathrm{vol}_M \, \mathrm{d}w \le C(l, \varepsilon) \int_M |\nabla u|,$$

where for $1 \le i \le s_l$ and $0 < \varepsilon < 1$

$$N_{i,\varepsilon}^l := \xi_i^l \left(B^l(0, 1 - \varepsilon)\right)$$

and $C(l,\varepsilon)$ is independent of u. As a consequence, by the Egorov inequality, there is a set $W_u^l \subset N_\varepsilon^l := N_{1,\varepsilon}^l \times \cdots \times N_{s_l,\varepsilon}^l \subset \mathbb{R}^{ls_l}$ for which

$$\begin{cases} \displaystyle\int_M |\nabla(p_w^l \circ u)| \le \frac{2C(l,\varepsilon)}{\mathcal{H}^{ls_l}(N_\varepsilon^l)} \int_M |\nabla u|, \quad \forall w \in W_u^l, \\ \mathcal{H}^{ls_l}(W_u^l) \ge \frac{1}{2}\mathcal{H}^{ls_l}(N_\varepsilon^l). \end{cases}$$

Meanwhile, observe that for fixed $w \in W_u^l$, the isomorphisms

$$\kappa_y := \gamma_\#\colon \pi_1(N, p_w^l(y)) \to \pi_1(N, y), \tag{3.2}$$

where $\gamma\colon [0,1] \to N$, $\gamma(0) = y$, $\gamma(1) = p_w^l(y)$ is any smooth curve, are independent of the choice of γ if its trajectory lies entirely in $(p_w^l)^{-1}(p_w^l(y))$. This is because any connected component of $(p_w^l)^{-1}(p_w^l(y))$ is simply-connected. Moreover, for any curve $\alpha\colon [0,1] \to N$, $\alpha(0) = \alpha(1) = y$, we have

$$\kappa_y \circ \chi\,([p_w^l \circ \alpha]) = [\alpha], \tag{3.3}$$

where χ is as in (3.1).

PROPOSITION 3.1. *Let u and $u_m \in R^\infty(M,N)$ be as above. Then, passing to a subsequence if necessary, there are $w_l \in N_\varepsilon^l$, $1 < l \le k$, such that*

(i) $u_m^{l-1} := p_{w_l}^l \circ u_m^l \in R^\infty(M, N^{l-1})$.

(ii) $\displaystyle\int_M |\nabla u_m^{l-1}|\,\mathrm{dvol}_M \le K(l,\varepsilon)\int_M |\nabla u_m^l|\,\mathrm{dvol}_M$.

(iii) *We have $\kappa_{u_m(x)} \circ \chi\,([u_m^1 \circ \alpha]) = [u_m \circ \alpha]$, where $\alpha\colon [0,1] \to M$, $\alpha(0) = \alpha(1)$, is any smooth curve avoiding the singularities of u_m^1.*

Regarding the above statements, the proof of this proposition is straightforward.

Step 2: Defining the inverse images which connect the singularities of u_m
Fix suitable $\varepsilon > 0$ and consider the sequence u_m^1 according to Proposition 3.1. Observe that $u_m^1 = \mathcal{P} \circ u_m$ where

$$\mathcal{P} := p_{w_2}^2 \circ \cdots \circ p_{w_k}^k.$$

Set

$$\tilde{u}_m := f \circ u_m^1\colon M \to \mathcal{S}_\beta.$$

f can be assumed to be smooth, so $\tilde{u}_m \in R^\infty(M, \mathcal{S}_\beta)$. Also, again by proposition 3.1, for some constant $C > 0$ independent of m

$$\int_M |\nabla u_m|\, d\,\mathrm{vol}_M \le C\int_M |\nabla u_m|\,\mathrm{d\,vol}_M.$$

We have then the following proposition:

PROPOSITION 3.2. *For $i = 1, \dots, \beta$, there is $y_i \in S_i^1$, $y_i \neq w$, a regular value of $f \circ \mathcal{P}$, such that y_i is a regular value for any $\tilde{u}_m$ and*

$$\mathcal{H}^{n-1}(\tilde{u}_m^{-1}(y_i)) \leq C' \int_M |\nabla u_m| \, \mathrm{d}\, \mathrm{vol}_M,$$

for $C' > 0$ independent of m.

Observe that we can project smoothly $\mathcal{S}_\beta$ on each of the circles $S_1^1, \dots, S_\beta^1$. The proof is straightforward, using Sard's theorem, the Egorov inequality and Theorem III.

Now observe that we can write

$$\tilde{u}_m^{-1}(y_i) = \bigcup_{j=1}^{\mu_i} A_m^{i,j} \subset M$$

and

$$(f \circ \mathcal{P})^{-1}(y_i) = \bigcup_{k=1}^{\nu_i} B^{i,k} \subset N,$$

where $A_m^{i,j}$ and $B^{i,k}$, respectively, the connected components of $\tilde{u}_m^{-1}(y_i)$ and $(\mathcal{P} \circ f)^{-1}(y_i)$, are smooth submanifolds of M and N. Moreover, it is obvious that $u_m(A_m^{i,j}) \subset B^{i,k}$ for some $1 \leq k \leq \nu_i$.

Using the isomorphisms κ_y defined above, we want to associate a unique, well-defined element of $\pi_1(N, y)$, $a_y^{i,k}$, to any $y \in B^{i,k}$. Since f is a homotopy equivalence, the $f^{-1}(y_i)$ are simply-connected. As a result, since $\mathcal{P}(B^{i,k}) \subset f^{-1}(y_i)$, the $B^{i,k}$ are simply-connected too (see (3.3)). Let $a^i \in \pi_1(\mathcal{S}_\beta, y_i)$ be the homotopy class representing the curves which make only one turn over S_i^1 in one fixed direction. Let $y' \in f^{-1}(y_i)$. Since f is a homotopy equivalence,

$$a_{y'}^i := (f_\#)^{-1}(a) \in \pi_1(N^1, y')$$

is well defined. We set for $y \in B^{i,k}$

$$a_y^{i,k} := k_y \circ \chi(a_{\mathcal{P}(y)}^i) \in \pi_1(N, y)$$

which is well defined by (3.2). Observe that by [7, section VII, theorem 7.2], for any $\gamma\colon [0, 1] \to B^{i,k}$ we have

$$\gamma_\#(a_{\gamma(1)}^{i,k}) = a_{\gamma(0)}^{i,k}. \tag{3.4}$$

Step 3: Modifying a map along the connecting sets

We should prove that any map $u_m \in R^\infty(M, N)$ can be approximated weakly by smooth maps with equibounded energy. Observe that to approximate a map in $W^{1,1}(M, N)$ by smooth maps we should take care of local and global obstructions as described in [16]. So what we will do is to approximate u_m by equibounded maps which satisfy the one-skeleton condition. This condition, introduced in [17], is the necessary and sufficient condition for that a map in $R^\infty(M, N)$ be strongly approximable by smooth maps. In fact, if for $u \in R^\infty(M, N)$, $u|_{M^1}$ is extendable to a smooth map $\tilde{u}\colon M \to N$ for every 'generic' 1-skeleton M^1 of M, then u can be approximated by smooth maps in $W^{1,1}(M, N)$ (see [17, theorem 6.2]).

DEFINITION 3.2. $u \in R^\infty(M, N)$ satisfies the 1-skeleton condition if and only if $u|_{M^1}$ is extendable to a smooth map $\tilde{u}\colon M \to N$ for every 'generic' 1-skeleton M^1 of M.

PROPOSITION 3.3. *Let u_m and $A_m^{i,j}$ as above. Then there are maps $v_{m,m'}$ such that the $v_{m,m'}$ satisfy the* 1*-skeleton condition and that*

$$\begin{cases} v_{m,m'} \xrightarrow{L^1} u_m \quad \text{as } m' \to \infty, \\[2ex] \displaystyle\int_M |\nabla v_{m,m'}| \le \int_M |\nabla u_m| + C \sum_{i=1}^{\beta} \sum_{j=1}^{\mu_i} \mathcal{H}^{n-1}(A_m^{i,j}) + O\left(\frac{1}{m'}\right) \end{cases}$$

for $C > 0$ independent of m.

For technical reasons we introduce a new version of lemma 5.1 in [18]:

LEMMA 3.1. *For every $1 \le i \le \beta$, and every $1 \le k \le \nu_i$, there exists an open covering of $B^{i,k}$, $\{U_1^{i,k}, \dots, U_{r_{i,k}}^{i,k}\}$, and smooth maps*

$$\omega_r^{i,k}\colon [0, 1] \times U_r^{i,k} \to B^{i,k}, \quad r = 1, \dots, r_{i,k}$$

such that

$$\begin{cases} \omega_r^{i,k}(0, y) = \omega_r^{i,k}(1, y) = y, \ \forall y \in B^{i,k}, \\[1ex] [\omega_r^{i,k}(., y)]_{\pi_p(N,y)} = a_y^{i,k}, \qquad \forall y \in B^{i,k}, \\[1ex] \displaystyle\int_0^1 |\nabla_x \omega_r^{i,k}(., y)|\, \mathrm{d}x \le C, \quad \forall y \in B^{i,k}, \\[1ex] |\nabla \omega_r^{i,k}|_\infty \le C, \end{cases}$$

where $C > 0$ is independent of i and k.

Using the compatibility condition (3.4) the proof of this lemma is straightforward. The maps $\omega_r^{i,k}$ are then used to introduce the $a_y^{i,k} \in \pi_1(N, y)$ in one-dimensional topological disks transversal to the inverse images $A_m^{i,j}$ and to modify the u_m. As in [18, lemma 7.1], the modified maps will satisfy the 1-skeleton condition with respect to M and N. This, alongside the energy bounds on $\omega_r^{i,k}$, completes the proof of the proposition.

Step 4: End of proof for Theorem I

If a map satisfies the 1-skeleton condition with respect to M and N, it can be approximated strongly in $W^{1,1}(M, N)$ by smooth maps. So we can consider the $v_{m,m'}$ to be smooth in Proposition 3.3. Remember that $\tilde{u}_m^{-1}(y_i)$ is the distinct union of the $A_m^{i,j}$. So, by Propositions 3.2 and 3.3, $v_{m,m}$ tend in L^1 to u and their gradients are equibounded in L^1 norm. By [11, vol. I, section 1.2.7]), $\nabla v_{m,m}$ converge in L^1 in the biting sense. Furthermore the limit cannot be other than ∇u, since $v_{m,m}$ converge strongly to u in L^1.

Acknowledgements

The author is grateful to Tristan Rivière for having drawn his attention to this problem and for the fruitful discussions we had about it. This research was carried out with support provided by the French government in the framework of cooperation programs between Université de Versailles and I.P.M., Institute for Studies in Theoretical Physics and Mathematics, Iran.

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