GAFA, Geom. funct. anal.
Vol. 13 (2003) 223 – 257
1016-443X/03/010223-35

GAFA Geometric And Functional Analysis

WEAK DENSITY OF SMOOTH MAPS FOR THE DIRICHLET ENERGY BETWEEN MANIFOLDS

M.R. Pakzad and T. Rivière

Abstract

We prove that given a simply connected compact manifold M and a closed manifold N, any map in the Sobolev space $W^{1,2}(M,N)$ can be approximated weakly by smooth maps between M and N.

1 Introduction

In the last decades many questions regarding the density of smooth maps in a given function space between manifolds have arisen in calculus of variations. Nowadays this kind of question is becoming a field on its own with widely open problems.

The most studied function spaces are the Sobolev spaces $W^{1,p}(M,N)$ where M is a compact n-dimensional manifold and N a closed Riemannian manifold isometrically embedded in some $\mathbb{R}^N$:

$$W^{1,p}(M,N) := \left\{u \in W^{1,p}(M,\mathbb{R}^N)\,;\ u(x) \in N \text{ a.e. } x \in M\right\}.$$

$W^{1,p}(M,\mathbb{R}^N)$ is defined on the base of the Sobolev space $W^{1,p}(\mathbb{R}^n,\mathbb{R}^N)$ using the smooth charts of M in the natural way. For any $\varphi \in C^\infty(\partial M,N)$, admitting a smooth extension $\phi : M \to N$, we define

$$C^\infty_\varphi(M,N) := \left\{u \in C^\infty(M,N)\,;\ u = \varphi \text{ on } \partial M\right\}$$

and

$$W^{1,p}_\varphi(M,N) := \left\{u \in W^{1,p}(M,N)\,;\ u = \varphi \text{ on } \partial M\right\}.$$

1.1 Local aspects of the sequentially weak density of smooth maps and the topological singular set. In [SU], [BZ], and [B1], respectively R. Schoen, K. Uhlenbeck, X. Zheng and F. Bethuel shed light on whether or not $C^\infty(\mathbf{B}^n,N)$ is dense in $W^{1,p}(\mathbf{B}^n,N)$, where $\mathbf{B}^n$ is the n-dimensional unit disk. They showed that the lack of approximability is due to the existence of a "topological singular set" for u. The singular set is characterized by local realizations of non-zero elements of $\pi_{[p]}(N)$ around points in $\mathbf{B}^n$ by u, where $[p]$ is the integer part of p. (The notion of topological singular set is still vague and remains to be defined precisely). In particular they proved that if $\pi_{[p]}(N) = 0$ then any map in $W^{1,p}(\mathbf{B}^n, N)$ can be approximated by smooth maps for the strong topology.

In the case $\pi_{[p]}(N) \neq 0$, the best that one can do is to approximate maps in $W^{1,p}(\mathbf{B}^n, N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth $(n-p-1)$-dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^{\infty,p}(\mathbf{B}^n, N)$. A map $v \in R^{\infty,p}(\mathbf{B}^n, N)$ realizes elements σ_x of $\pi_{[p]}(N)$ on the $[p]$-spheres centered at any point $x \in \Sigma(v)$ and contained in the normal $[p]+1$ plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is non-trivial, then v cannot be approximated by smooth maps in the strong topology (see [B1]). Furthermore one can assign to v a $\pi_{[p]}(N)$-chain which is carried by $\Sigma(v)$ with "multiplicity" σ_x at each point x of $\Sigma(v)$. This $\pi_{[p]}(N)$-chain can be called the topological singular set $\mathbf{S}_v$ of v in $R^{\infty,p}(\mathbf{B}^n, N)$. One of the major questions would be to understand the behavior of $\mathbf{S}_{v_m}$ for a sequence of maps $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ converging to any $u \in W^{1,p}(\mathbf{B}^n, N)$ and eventually to prove a "flat-norm" convergence of $\mathbf{S}_{v_m}$ to a unique flat $\pi_{[p]}(N)$-chain $\mathbf{S}_u$ which therefore we could call the topological singular set of u.

Related to this question is the problem of the weak density of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$. Although the density of smooth maps for the weak topology can be easily handled (see [B1]: Smooth maps are dense for the weak topology if and only if $p \in \mathbb{N}$), the question of the density of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$ for the *sequentially* weak topology, where $p \in \mathbb{N}$, is more involved: *For $p \in \mathbb{N}$, $\pi_p(N) \neq 0$, does there exist for any $u \in W^{1,p}(\mathbf{B}^n, N)$ a sequence $u_m \in C^\infty(\mathbf{B}^n, N)$ such that $u_m \rightharpoonup u$ in $W^{1,p}$?*

The case $N{=}S^2$, $p{=}2$ was treated by F. Bethuel, H. Brezis, J.M. Coron and E. Lieb in [BrCL] and [B2]. Bethuel mentioned that the answer is yes for $N = S^p$, $p \geq 2$ [B1]. In [H], P. Hajlasz has proved that the answer is yes when N is $(p-1)$-connected. For other positive answers to the sequentially weak density of smooth maps in other situations see [P3] and [R]. Counterexamples to the above question are not known.

As we will explain below the control of the mass of the minimal chain connecting $\mathbf{S}_{v_m}$ for $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ converging strongly to u enables us to give a positive answer to the sequentially weak density of smooth maps in the case where $M = \mathbf{B}^n$.

1.2 The global problem. Recent developments by F. Hang and F.H. Lin [HaL1] showed that one should be careful while considering a generic smooth compact manifold M as the domain. In particular there are cases where the condition "$\mathbf{S}_u = 0$" is not sufficient to guarantee the approximability of u by smooth maps in the strong topology of $W^{1,p}(M,N)$, even when $N = S^p$. As an example, consider the map $u : \mathbb{CP}^2 \to S^2$ defined by

$$u\big([x,y,z]\big) := [x,y]\,,$$

where $[x,y,z]$ (respectively $[x,y]$) are the homogeneous coordinates on $\mathbb{CP}^2$ (respectively on $\mathbb{CP}^1 = S^2$). The map $u \in W^{1,2}(\mathbb{CP}^2, S^2)$ has only a point singularity on $a = [0,0,1]$, hence $\mathbf{S}_u = 0$. However, u is not in the strong closure of smooth maps (see [HaL1]). Moreover, this *global* singularity cannot be "located" in the domain, in the sense that we may approximate strongly u by maps which are smooth on a fixed neighborhood of the point a and which are singular in another point.

We are able to handle both local obstructions $\mathbf{S}_u$ and global ones to the strong approximation in order to establish smooth weak approximability whenever $p = 2$ and M is simply connected. Precisely, our main result is

Theorem I. *Let M and N be compact smooth manifolds and assume that M is simply connected. Then smooth maps are dense in $W^{1,2}(M,N)$ for the sequential weak topology. Moreover, assuming that $\varphi \in C^\infty(\partial M, N)$ is smoothly extendable to M, for every $u \in W^{1,2}_\varphi(M,N)$, there is a sequence of smooth maps $u_m \in C^\infty_\varphi(M,N)$ converging weakly to u in $W^{1,2}$.* □

We will show in the last section of this paper how we can manage to remove the global singularities for getting these weak sequential density results even if the domain is not a disk. We will use recent results of F. Hang and F.H. Lin, appeared in [HaL2], which give the necessary and sufficient condition for a map in $W^{1,p}(M,N)$ to be strongly-approximable by smooth maps in this space. The idea would be to modify the maps, not along the minimal connections, but along inverse images. We will show how this operation produces a new map which satisfies the Hang–Lin condition.

1.3 More results on the topological singular sets $\mathbf{S}_u$ and sequentially weak density. In this paper, we prove the convergence of the $\pi_{[p]}(N)$-chains $\mathbf{S}_{v_m}$ for any convergent sequence of maps in $W^{1,p}(\mathbf{B}^n, N)$ when $[p] = n-1$ if N is $([p]-1)$-connected, i.e.

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0$$

or when $[p] = 1$ if $\pi_1(N)$ is abelian. The problem is still open for almost every other value for $[p]$. In fact, if we set for $\mathbf{S}$, any integral flat chain in $\mathbf{B}^n$ of dimension k,

$$m_i(\mathbf{S}) := \inf\big\{\mathbf{M}(\mathbf{T})\,;\ T \in \mathcal{R}_{k+1}(\mathbf{B}^n),\, \partial\mathbf{T} = \mathbf{S}_u\big\}\,,$$

the question would be to determine whether $m_i(\mathbf{S}_{u_m} - \mathbf{S}_{u_k}) \to 0$ when u_m converges strongly to u in $W^{1,p}(\mathbf{B}^n, S^p)$. The answer is yes for $p = 1$ or $n-1$ (see [BBC] and [GMS, vol. II, sect. 5.4.2]), while we do not know whether this is the case even for maps in $H^1(\mathbf{B}^4, S^2)$.

Theorem II. *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$. Assume that $[p] = 1$ and $\pi_1(N)$ is abelian or $[p] = n-1$ and N is a closed $([p]-1)$-connected Riemannian manifold of dimension equal or greater than $[p]$. Then $\mathbf{S}_u$, the topological singular set of any $u \in W^{1,p}(\mathbf{B}^n, N)$, is well defined as a flat $\pi_{[p]}(N)$-chain and the flat norm of $\mathbf{S}_{u_m} - \mathbf{S}_u$ converges to 0 if $u_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$. Moreover $u \in W^{1,p}(\mathbf{B}^n, N)$ (respectively $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$) is a strong limit of maps in $C^\infty(\mathbf{B}^n, N)$ (respectively $C^\infty_\varphi(\mathbf{B}^n, N)$) if and only if $\mathbf{S}_u = 0$.* □

Remark 1.1. The approach used in ([GMS, vol. II, sect. 5.4.2]) for defining a topological singularity for Sobolev maps considers only the real homologic singularities. This is not suitable when the homotopy type singularities are not seen by the real homology, as in the case $W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ discussed below.

Remark 1.2. We can extend these results to $[p] = 3$ or 7 [P2].

However, our method allows us to prove the following theorem. This result is not mentioned by Hajlasz [H] and cannot be deduced directly from his proof.

Theorem III. *Let N be a closed smooth manifold. Assume that for some integer $2 \le p \le k$, N is $(p-1)$-connected. Also assume that $\varphi : \partial\mathbf{B}^n \to N$ is a smooth map, smoothly extendable to $\mathbf{B}^n$. Then for every $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ such that u_m converge weakly to u in $W^{1,p}_\varphi(\mathbf{B}^n, N)$.* □

If $p = 1$, smooth maps are not sequentially dense in $W^{1,1}(\mathbf{B}^n, N)$ for most cases [Ha]. Meanwhile, assuming that $\pi_1(N)$ is abelian, by controlling the mass of connections for a convergent sequence in $W^{1,1}(\mathbf{B}^n, N)$, a weaker type of density is obtained. The non-abelian case is more involved and is treated in another paper [P3].

Definition 1.1. *Let Ω be a domain in $\mathbb{R}^n$ and let u_m be a bounded sequence in $L^1(\Omega)$. u_m is said to converge in the biting sense to $u \in L^1(\Omega)$ if for every $\varepsilon > 0$ there exists a measurable set $E \subset \Omega$ such that $\mu(E) < \varepsilon$ and $u_m \rightharpoonup u$ weakly in $L^1(\Omega \backslash E)$.*

Theorem IV. *Let $\mathbf{B}^n$ be the unit disk in $\mathbb{R}^n$ and N be any k-dimensional closed manifold. Assume that $\varphi \in C^\infty(\partial\mathbf{B}^n, N)$ is smoothly extendable to $\mathbf{B}^n$. If $\pi_1(N)$ is abelian, for every $u \in W^{1,1}_\varphi(\mathbf{B}^n, N)$ there is a sequence of maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, N)$ such that ∇u_m tend to ∇u in the biting sense.*□

2 Preliminaries

2.1 Flat chains over a coefficient group. Let G be an abelian group. $|\,.\,| : G \to \mathbb{R}^+$ is called a norm on G if

$$\begin{aligned} (i) &\quad \forall g \in G\,, \quad |-g| = |g|\,, \\ (ii) &\quad \forall g,h \in G\,, \quad |g+h| \le |g| + |h|\,, \\ (iii) &\quad |g| = 0 \text{ if and only if } g = 0\,. \end{aligned}$$

We assume that G is a complete metric space with respect to the metric $d(g,h) := |g-h|$.

Let K be any compact convex subset of $\mathbb{R}^n$. We introduce the spaces of polyhedral k-chains, flat k-chains and finite mass flat k-chains in K, with coefficients in G. The readers can refer to [Fl] and [W] for more details.

Definition 2.1. *$\mathcal{P}_k(K,G)$ is the space of all G-linear sums of oriented k-dimensional polyhedra in K. For $P = \sum_{i=1}^m g_i[[\sigma_i]] \in \mathcal{P}_k(K,G)$, where $g_i \in G$ and σ_i, $i = 1,\dots,m$, are non-overlapping k-dimensional polyhedra, we define the mass and the boundary of P respectively to be*

$$\mathbf{M}(P) := \sum_{i=1}^m |g_i| \operatorname{vol}(\sigma_i)\,,$$

$$\partial P := \sum_{i=1}^m g_i\, \partial[[\sigma_i]] \in \mathcal{P}_{k-1}(K,G)\,.$$

Definition 2.2. *Let $P \in \mathcal{P}_k$ be a polyhedral G-chain. The flat norm of P is*

$$\mathcal{F}(P) := \inf\left\{\mathbf{M}(P - \partial B) + \mathbf{M}(B)\,;\ B \in \mathcal{P}_{k+1}\right\}.$$

Definition 2.3. *The space of flat k-chains, $\mathcal{F}_k(K,G)$, is the $\mathcal{F}$-completion of $\mathcal{P}_k(K,G)$. For $A \in \mathcal{F}_k(K,G)$, we define the mass of A to be*

$$\mathbf{M}(A) := \inf\left\{\liminf_{n\to\infty} \mathbf{M}(P_n)\,;\ P_n \xrightarrow{\mathcal{F}} A,\ P_n \in \mathcal{P}_k(K,G)\right\}.$$

$\mathcal{M}_k(K,G)$ is the set of flat k-chains in $\mathcal{F}_k(K,G)$ with finite mass and is a complete metric space with respect to the flat norm. Finally, for Ω being any open set in $\mathbb{R}^n$, we define $\mathcal{F}_k(\Omega,G)$ to be the union of all the $\mathcal{F}_k(K,G)$ among convex compact sets $K \subset \Omega$.

We recall some useful results:

LEMMA 2.1. *The boundary map $\partial : \mathcal{P}_k \to \mathcal{P}_{k-1}$ is continuous with respect to the $\mathcal{F}$-norm and so it can be extended to a unique $\mathcal{F}$-continuous map $\partial : \mathcal{F}_k \to \mathcal{F}_{k-1}$.* □

LEMMA 2.2. *Any homomorphism $\chi : G \to H$ between groups, which is continuous with respect to their norms, induces a $\mathcal{F}$-continuous group homomorphism*

$$\chi_* : \mathcal{F}_k(K,G) \to \mathcal{F}_k(K,H)\,.$$

Moreover, χ_* commutes with ∂, i.e.

$$\chi_*(\partial A) = \partial \chi_*(A)\,, \quad \forall A \in \mathcal{F}_k(K,G) \tag{2.1}$$

and

$$\mathbf{M}(\chi_*(A)) \le C\mathbf{M}(A), \quad \forall A \in \mathcal{M}_k(K,G)$$

if $|\chi(g)| \le C|g|$ for all $g \in G$. □

2.2 The subspaces $\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ and $R^{\infty,p}(\mathcal{C}^n, N)$.

DEFINITION 2.4. *Let $\mathcal{C}^n := [-1/2, 1/2]^n$ be the unit cube in $\mathbb{R}^n$. $u \in W^{1,p}(\mathcal{C}^n, N)$ is in $\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ if u is smooth except on $\Sigma(u) = \sum_{i=1}^r \Sigma_i$, where for $i = 1, \dots, r$, Σ_i is a subset of a linear subspace of $\mathbb{R}^n$ of dimension $n-p-1$ and $\partial\Sigma_i$ is a subset of a linear subspace of dimension $n-p-2$.*

Theorem (Bethuel, [B1]). *$\mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ (respectively $\mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$) is dense in $W^{1,p}(\mathcal{C}^n, N)$ (respectively $W_\varphi^{1,p}(\mathcal{C}^n, N)$) for the strong topology.* □

Let $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$. There is some compact subset of $\mathcal{C}^n$, $B = \bigcup_{i=1}^\mu \sigma_i$, where the σ_i, $i = 1, \dots, \mu$ are non-overlapping $(n-p-1)$-dimensional polyhedra, such that $\Sigma(u) \subset B$ and that every $n-p-2$ dimensional face of B belongs to at least two σ_i. Moreover we can assume that any two different faces of B intersect only on their boundaries. Let

$$\|x\| := \max_{i=1,\dots,n} |x_i| \quad \text{for } x = (x_1, \dots, x_n) \in \mathbb{R}^n$$

and for $\delta > 0$ put

$$V^\delta := \left\{ y \in \mathcal{C}^n\,;\ \|y - B\| \le \delta \right\}$$

where

$$\|y - B\| := \inf \left\{ \|y - x\|\,;\ x \in B \right\}.$$

Also for $\delta > 0$ and some orthonormal base $\{e_1^i, \dots, e_{p+1}^i\}$ orthogonal to σ_i, set

$$\sigma_i^\delta := \left\{ x + \sum_{j=1}^{p+1} t_j e_j^i\,;\ x \in \sigma_i\,,\ \max_{j=1,\dots,p+1} |t_j| \le \delta \right\}$$

and define $\pi_i : \sigma_i^\delta \to \sigma_i$ to be the smooth projection

$$\pi_i\left(x + \sum_{j=1}^{p+1} t_j e_j^i\right) := x\,.$$

For δ_0 small enough, we consider a lipschitz projection $\pi : V^{\delta_0} \to B$ with the following properties:

(i) $V^\delta = \bigcup_{i=1}^{\mu} V_i^\delta$, where the $V_i^\delta := \pi^{-1}(\sigma_i) \cap V^\delta$ are non-overlapping n-polyhedra in $\mathbb{R}^n$ which intersect only on lower dimensional faces.

(ii) There are lipschitz diffeomorphisms

$$f_i : V_i^{\delta_0} \to \sigma_i^{\delta_0},$$

such that

$$\begin{cases} f_i(V_i^\delta) = \sigma_i^\delta & \forall \delta < \delta_0 \\ \pi|_{V_i^\delta} = \pi_i \circ f_i|_{V_i^\delta} & \\ f_i\big([x, \pi(x)]\big) = \big[f_i(x), \pi(x)\big] & \forall x \in V_i^\delta \end{cases}$$

where by $[p, q]$ we mean the segment joining the two points in $\mathcal{C}^n$.

Definition 2.5. *For $y \in V^\delta \backslash B$, let $h_\delta(y)$ be the unique point on ∂V^δ which is on the ray from $\pi(y)$ to y. Then naturally $\pi(h_\delta(y)) = \pi(y)$ and h_δ is locally lipschitz on $V^\delta \backslash B$. We set*

$$u_\delta(y) := \begin{cases} u(h_\delta(y)) & \text{if } y \in V^\delta\,, \\ u(y) & \text{otherwise}\,. \end{cases} \tag{2.2}$$

Definition 2.6. *We set*

$$R^{\infty,p}(\mathcal{C}^n, N) := \big\{u_\delta\,;\ u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)\big\}$$

and we say u is radial if $u \in R^{\infty,p}(\mathcal{C}^n, N)$.

By computing the integral of u_δ on V_i^δ by the mean of f_i as new coordinates we observe that for $\delta_1 > 0$ sufficiently small, there is some constant K, depending only on B, for which

$$\begin{cases} \int_{\partial V^\delta} |\nabla u|^p \le \frac{K}{\delta_1} \int_{V^{\delta_1}} |\nabla u|^p \\ \int_{V^\delta} |\nabla u_\delta|^p \le \delta K \int_{\partial V^\delta} |\nabla u|^p \end{cases} \tag{2.3}$$

for $\delta \in I_0$, a positive measure subset of $[0, \delta_1]$.

Remark 2.1. As a result, $R^{\infty,p}(\mathcal{C}^n, N)$ is also dense in $W^{1,p}(\mathcal{C}^n, N)$ for the strong topology.

We recall that there are canonical isomorphisms between $\pi_p(N, x)$ and $\pi_p(N, y)$ for $x, y \in N$ if and only if $\pi_1(N)$ is abelian for $p = 1$ and $\pi_1(N) = 0$ for $p > 1$. We assume that these conditions are satisfied so that we can talk about the homotopy classes of maps from S^p into N as elements of $\pi_p(N)$.

DEFINITION 2.7. *Let $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ and $\Sigma(u) \subset B = \bigcup_{i=1}^{\mu} \sigma_i$ be its singular set. Assume that each σ_i is oriented by a smooth $(n-p-1)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$, let N_a be the $(p+1)$-dimensional plane orthogonal to σ at a. Consider the $(p+1)$-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the $(p+1)$-vector $\vec{M}_a$ such that $\vec{\sigma}_i(a) \wedge \vec{M}_a = \xi_{\mathbb{R}^n}$. u is continuous on the p-dimensional oriented sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$. The homotopic singularity of u at σ_i is*

$$[u, \sigma_i] := [u|_{\Sigma_{a,\delta}}]_{\pi_p(N)}\,, \tag{2.4}$$

i.e. the homotopy class of $u|_{\Sigma_{a,\delta}}$ in $\pi_p(N)$, which is independent of the choices of a and δ.

DEFINITION 2.8. *We define the topological singularity of $u \in \mathcal{R}^{\infty,p}(\mathcal{C}^n, N)$ to be the $\pi_p(N)$-polyhedral chain*

$$\mathbf{S}_u := \sum_{i=1}^{\mu} [u, \sigma_i]\,[[\sigma_i]] \in \mathcal{P}_{n-p-1}\big(\mathcal{C}^n, \pi_p(N)\big)\,,$$

where $\Sigma(u) \subset B = \bigcup_{i=1}^{\mu} \sigma_i$ is its singular set.

REMARK 2.2. u suffices to be continuous on $\mathcal{C}^n \backslash B$ for $\mathbf{S}_u$ to be well defined.

2.3 A useful lemma. Let $\mathbf{B}^l$ be the unit disk in $\mathbb{R}^l$. We denote

$$U^l := \big\{(x,y) \in \mathbf{B}^l \times \mathbf{B}^l\,;\ x \neq y\big\}$$

and

$$U^l_\delta := \big\{(x,y) \in U^l\,;\ y \notin B(x,\delta)\big\}\,.$$

DEFINITION 2.9. *For $(x,y) \in U^l$, we define $p(x,y)$ to be the unique point on $\partial \mathbf{B}^l$ which is on the ray from x to y.*

Clearly p is well-defined and smooth on U^l. As U^l_δ is compact, we have for some constant $C(l,\delta) > 0$:

$$\sup_{(x,y)\in U^l_\delta} \big|\nabla p(x,y)\big| \leq C(l,\delta) < +\infty\,.$$

We have

LEMMA 2.3. *Let $1 \leq p < l$ be an integer. Then*

$$\int_{B(0,1-\delta)} \big|\nabla_y p(x,y)\big|^p dx \leq C(l,p,\delta) \tag{2.5}$$

when $C(l,p,\delta)$ depends on l, p and δ and not on y. □

Proof. Let $x \in B(0,1-\delta)$. We distinguish two cases:

(i) $y \notin B(x,\delta)$. Then $(x,y) \in U^l_\delta$ and we get

$$\big|\nabla_y p(x,y)\big| \leq C(l,\delta) \leq \frac{2C(l,\delta)}{|y-x|}\,. \tag{2.6}$$

(ii) Otherwise $y \in B(x,\delta) \subset \mathbf{B}^l$. Then

$$p(x,y) = p\left(\frac{y-x}{|y-x|}\delta + x, x\right)$$

and so

$$|\nabla_y p(x,y)| \le \left|\nabla_y p\left(\frac{y-x}{|y-x|}\delta + x, x\right)\right| \frac{\delta l}{|y-x|} \le \frac{\delta l C(l,\delta)}{|y-x|}\,, \tag{2.7}$$

as

$$\left(\frac{y-x}{|y-x|}\delta + x, x\right) \in U_\delta^l\,.$$

Using the inequalities (2.6) and (2.7), the lemma is proved. □

3 An Example: $W^{1,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$

3.1 Notation. Let $f : S^2 \to \mathbb{R}^6$ be the map

$$f(x,y,z) := \left(\tfrac{\sqrt{2}}{2}x^2, \tfrac{\sqrt{2}}{2}y^2, \tfrac{\sqrt{2}}{2}z^2, xy, yz, zx\right). \tag{3.1}$$

f induces an embedding of the 2-dimensional Real Projective Space, $\mathbb{RP}^2$, into $\mathbb{R}^6$. A property of this embedding is that the minimum length of the cycle homotopic to the non-zero element of $\pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ is π, independent of the choice of the base point. We define a norm on the 2-group $\pi_1(\mathbb{RP}^2)$:

$$|a| := 1 \quad \text{if} \quad a \neq 0\,, \ := 0 \quad \text{otherwise}\,. \tag{3.2}$$

Also we define the map $g : \mathbf{B}^2 \to \mathbb{RP}^2$ as follows:

$$g(x_1,x_2) := f\big(x_1, x_2, \sqrt{1-(x_1^2+x_2^2)}\big)\,. \tag{3.3}$$

Now let $w_0 = f(1,0,0) \in \mathbb{RP}^2$ and put

$$\mathcal{G} = f\big(\{(x,y,z) \in S^2\,;\ z = 0\}\big)\,.$$

$\mathcal{G}$ is a length minimizing generator of $\pi_1(\mathbb{RP}^2)$ passing through w_0. For $w \in \mathbb{RP}^2\backslash\mathcal{G}$ we define the projection

$$p_w : \mathbb{RP}^2\backslash\{w\} \to \mathcal{G}$$

as follows:

$$p_w(w') := g\big(p(g^{-1}(w), g^{-1}(w'))\big) \quad \forall w' \in \mathbb{RP}^2\backslash\{w\} \tag{3.4}$$

where p is the map given in Definition 2.9. Observe that p_w is well defined for $w' \in \mathcal{G}$ as in this case we would have $p_w(w') = w'$ independent of the choice of $g^{-1}(w')$. Let us fix $\varepsilon > 0$ such that

$$\mathrm{Vol}(\mathbb{RP}^2\backslash\mathcal{G}_\varepsilon) > 2\pi\,,$$

where

$$\mathcal{G}_\varepsilon := \big\{y \in \mathbb{RP}^2\,;\ d(y,\mathcal{G}) < \varepsilon\big\}$$

is the ε-neighborhood of $\mathcal{G}$ in $\mathbb{RP}^2$.

LEMMA 3.1. *Let $\mathcal{G}$ and p_w be as above. Then*

(i) $p_w : \mathbb{RP}^2\backslash\{w\} \to \mathcal{G}$ *is well defined and smooth.*

(ii) For any cycle $\mathcal{G}' \subset \mathbb{RP}^2\backslash\{w\}$ *we have*

$$[\mathcal{G}']_{\pi_1(\mathbb{RP}^2)} = \chi\big([p_w(\mathcal{G}')]_{\pi_1(\mathcal{G})}\big) \tag{3.5}$$

where $\chi : \pi_1(\mathcal{G}) \simeq \mathbb{Z} \to \pi_1(\mathbb{RP}^2) \simeq \mathbb{Z}_2$ *is an onto homomorphism.*

(iii) For any $w' \in \mathbb{RP}^2$ *we have*

$$\int_{\mathbb{RP}^2\backslash\mathcal{G}_\varepsilon} \big|\nabla p_w(w')\big| dw = C_0 = C_0(\varepsilon) < +\infty\,. \tag{3.6}$$

□

Proof. We observe that g^{-1} is well defined and smooth on $\mathbb{RP}^2\backslash\mathcal{G}$, while in a neighborhood of $\mathcal{G}$, p_w is a projection along smooth curves orthogonal to $\mathcal{G}$. This proves the first part of the lemma. Now observe that the injection map $i : \mathcal{G} \to \mathbb{RP}^2$ induces a homomorphism,

$$\chi : \pi_1(\mathcal{G}) \to \pi_1(\mathbb{RP}^2)\,,$$

which is onto as $[\mathcal{G}]$ is the generator of $\pi_1(\mathbb{RP}^2)$. So, since p_w is smooth on $\mathbb{RP}^2\backslash\{w\}$, we get

$$[\mathcal{G}']_{\pi_1(\mathbb{RP}^2)} = [p_w(\mathcal{G}')]_{\pi_1(\mathbb{RP}^2)} = \chi\big([p_w(\mathcal{G}')]_{\pi_1(\mathcal{G})}\big)$$

which proves (3.5).

Now let

$$N_\varepsilon := \mathbb{RP}^2\backslash\mathcal{G}_\varepsilon$$

and observe that for $g : \mathbf{B}^2 \to \mathbb{RP}^2$ as in (3.3)

(i) $N_{\varepsilon/2} = g(B(0, 1-\delta))$, for some $0 < \delta < 1$,

(ii) $g|_{B(0,1-\delta)}$ is an embedding.

We prove (3.6). Let $w \in N_\varepsilon \subset N_{\varepsilon/2}$. If $w' \notin \mathcal{G}_{\varepsilon/2}$ then since g^{-1} is smooth on $N_{\varepsilon/2}$, using (2.5) and (3.4), we get, for some $C_0(\delta) > 0$,

$$\int_{N_\varepsilon} \big|\nabla p_w(w')\big| dw \le \int_{N_{\varepsilon/2}} \big|\nabla p_w(w')\big| dw \le C_0(\delta)\,.$$

If not, the map $\tilde{p} : N_\varepsilon \times \overline{\mathcal{G}}_{\varepsilon/2} \to \mathcal{G}$

$$\tilde{p}(w, w') := p_w(w')$$

is smooth on its compact domain because $N_\varepsilon \cap \overline{\mathcal{G}}_{\varepsilon/2} = \emptyset$. So there exists $K > 0$, independent of w, w' for which

$$\big|\nabla p_w(w')\big| \le K$$

if $w' \in \mathcal{G}_{\varepsilon/2}$, $w \in N_\varepsilon$. This completes the proof of (3.6). □

3.2 Study of $R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$. Let $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$. We observe that $\mathbf{S}_u \in \mathcal{P}_0(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is is in fact the sum $\sum_{i=1}^{\mu}[u,p_i][[p_i]]$ where $\{p_1,\dots,p_\mu\}$ are the singularities of u and $[u,p_i]$ is the class of $u(\partial B(p_i,\delta))$ in $\pi_1(\mathbb{RP}^2)$ for δ small enough.

DEFINITION 3.1. *$I \in \mathcal{F}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is a connection for u if $\partial I = \mathbf{S}_u$.*

PROPOSITION 3.1. *For $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$, there exists $I \in \mathcal{P}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ such that*

$$\begin{cases} \partial I = \mathbf{S}_u \\ \mathbf{M}(I) \le C \int |\nabla u| + C \end{cases} \tag{3.7}$$

for some constant $C > 0$ depending only on φ. □

REMARK 3.1. Any $I \in \mathcal{P}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ is a set of non-oriented segments while $\mathbf{M}(I)$ is simply the total length of these segments.

COROLLARY 3.1. *For any $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$, there exists a connection $I_u \in \mathcal{F}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ of minimal mass which satisfies*

$$\mathbf{M}(I_u) \le C \int |\nabla u| + C\,.$$

(Use the compactness result of [F1, section 4.2.26, p. 432].) □

Proof of Proposition 3.1. First we assume that $\varphi \equiv w_0$ is constant. Let u be a map in $R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$ for which $\mathbf{S}_u = \sum_{i=1}^{\mu}[u,p_i][[p_i]]$. Let A be the set of regular values of u in $\mathbb{RP}^2$. By Sard's theorem, $\mathcal{H}^2(A) = \mathrm{vol}(\mathbb{RP}^2) = 4\pi$. We estimate the integral

$$J := \int_{\mathbb{RP}^2\setminus\mathcal{G}_\varepsilon} \int_{\mathcal{C}^2} \big|\nabla(p_w \circ u)(x)\big| dx\, dw\,. \tag{3.8}$$

We have, by (3.6),

$$J \le \int_{\mathcal{C}^2} \int_{\mathbb{RP}^2\setminus\mathcal{G}_\varepsilon} \big|\nabla p_w(u(x))\big|\big|\nabla u(x)\big| dw\, dx \le C_0 \int_{\mathcal{C}^2} |\nabla u|\,.$$

As a result, considering (3.8), there exists some positive measure set $W \subset \mathbb{RP}^2\setminus\mathcal{G}_\varepsilon$ such that

$$\int_{\mathcal{C}^2} \big|\nabla(p_w \circ u)\big| \le \frac{C_0}{2\pi} \int_{\mathcal{C}^2} |\nabla u| \quad \text{for all } w \in W\,. \tag{3.9}$$

Since u is *radial*, for some regular $w \in A \cap W$, $u^{-1}(w)$ is a finite subset of $\mathcal{C}^2$. We have

LEMMA 3.2. *There exists $w \in W$ such that the map*

$$\tilde{u} := p_w \circ u : \mathcal{C}^2 \to \mathcal{G}$$

is in $\mathcal{R}^{\infty,1}_{w_0}(\mathcal{C}^2,\mathcal{G})$. Moreover if we consider the additive group $\mathbb{Z}$ with its usual norm, for some $\tilde{I} \in \mathcal{P}_1(\mathcal{C}^2,\mathbb{Z})$, for which $\partial\tilde{I} = \mathbf{S}_{\tilde{u}}$, the following properties hold:

$$\begin{cases} L(\tilde{I}) = \inf\left\{L(\tilde{I}')\,;\ \tilde{I}' \in \mathcal{F}_1(\mathcal{C}^2,\mathbb{Z})\,,\ \partial\tilde{I}' = \mathbf{S}_{\tilde{u}}\right\} \\ L(\tilde{I}) \le \frac{1}{\pi}\int_{\mathcal{C}^2}|\nabla\tilde{u}|\,. \end{cases} \tag{3.10}$$

where $L(\tilde{I})$ is the $\mathbb{Z}$-mass of $\tilde{I}$. □

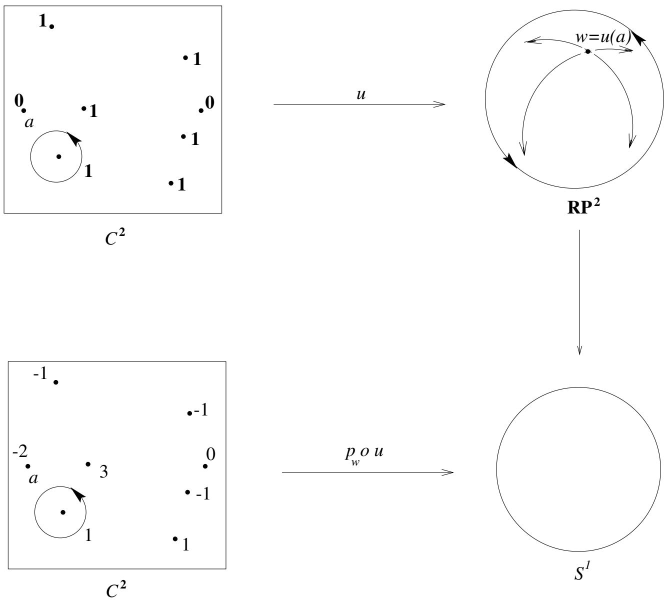

Figure 1: Projection of u into S^1

Remark 3.2. Observe that $\pi_1(\mathcal{G}) \simeq \mathbb{Z}$. Moreover $L(\tilde{I})$ is the length of minimal connections connecting the singularities of $\tilde{u}$, introduced in [BrCL].

For a proof of this lemma, see [DH, Propositions 1 and 2]. Observe that the best constant in inequality (3.10) is achieved by the mean of co-area formula as in [ABL].

Using Lemma 3.2, we finish the proof of the proposition: Consider the homomorphism χ in (3.5). By Lemma 2.2, χ induces a group homomorphism

$$\chi_* : \mathcal{P}_k(\mathcal{C}^2,\mathbb{Z}) \to \mathcal{P}_k\big(\mathcal{C}^2,\pi_1(\mathbb{RP}^2)\big)\,.$$

We consider $\tilde{I}$ as in Lemma 3.2 and we set $I := \chi_*(\tilde{I})$. We deduce that

$$\partial I = \chi_*(\mathbf{S}_{\tilde{u}})\,. \tag{3.11}$$

Meanwhile, by Lemma 3.1, part (ii), we observe that, for all points $p \in \mathcal{C}^2$, there exists δ small enough for which
$$[u,p] = \big[u(\partial B_\delta(p))\big]_{\pi_1(\mathbb{RP}^2)} = \chi\big([\tilde{u}(\partial B_\delta(p))]_{\pi_1(\mathcal{G})}\big) = \chi\big([\tilde{u},p]\big)$$
and as a result
$$\mathbf{S}_u = \chi_*(\mathbf{S}_{\tilde{u}})\,. \tag{3.12}$$

For visualizing this phenomenon see Fig. 1 where we compare the singularities of u and $p_w \circ u$. Comparing this with (3.11) we obtain
$$\partial I = \mathbf{S}_u\,.$$
Observe that $|\chi(z)| \le |z|$ for all $z \in \mathbb{Z}$, thus we have by Lemma 3.2
$$\mathbf{M}(I) = \mathbf{M}(\chi_*(\tilde{I})) \le L(\tilde{I}) \le \frac{1}{\pi}\int_{\mathcal{C}^2} \big|\nabla(p_w \circ u)\big|\,.$$
So using the inequality (3.9), we get
$$\mathbf{M}(I) \le \frac{C_0}{2\pi^2}\int_{\mathcal{C}^2} |\nabla u|\,.$$
This completes the proof for constant boundary data. In Fig. 2 we have illustrated two connections for u and one for $p_w \circ u$. We show how the minimal polyhedral connection for u (the thin dashed segments) comes to be lesser in mass from the image of any connection of $p_w \circ u$ under χ_* (the thick curves).

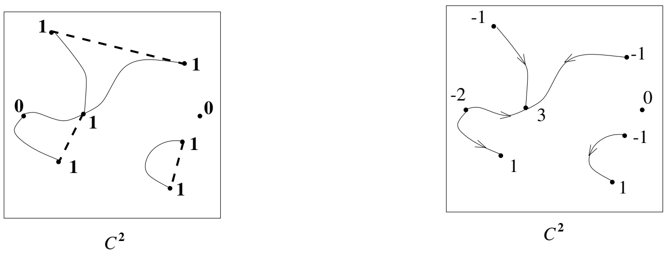

Figure 2: Connections for u and for $p_w \circ u$

Now consider the case of non-constant φ. We extend u over the cube
$$\widetilde{\mathcal{C}}^2 := \big\{x \in \mathbb{R}^2\,;\ \|x\| \le \tfrac{1}{2} + \varepsilon\big\}$$
for some $\varepsilon > 0$ as follows:
$$u(x) := \phi\left(\frac{1/2 + \varepsilon - \|x\|}{\varepsilon}x\right) \qquad \forall x \in \widetilde{\mathcal{C}}^2\backslash\mathcal{C}^2\,,$$
while ϕ is the smooth extension of φ onto $\mathcal{C}^2$. Now u is constant on the boundary of $\widetilde{\mathcal{C}}^2$ and we have clearly
$$\int_{\widetilde{\mathcal{C}}^2} |\nabla u| \le \int_{\mathcal{C}^2} |\nabla u| + C_1$$

where C_1 depends only on φ. Applying the proposition to u on $\widetilde{\mathcal{C}}^2$ as above, we obtain some $I' \in \mathcal{P}_1(\widetilde{\mathcal{C}}^2, \mathbb{Z}_2)$ for which $\partial I' = \mathbf{S}_u$ and $M(I') \leq CE(u)+C$. Now since spt $\mathbf{S}_u$ is a compact set in $\mathcal{C}^2$, we observe that there is an open $U \subset \mathcal{C}^2$ such that spt $\mathbf{S}_u \subset U$ and ∂U is a convex polygon. Let Π denote the lipschitz map which leaves U fixed and radially projects points outside U onto its boundary. This map induces a map

$$\Pi_{\#} : \mathcal{P}_k(\widetilde{\mathcal{C}}^2, \mathbb{Z}_2) \to \mathcal{P}_k(\mathcal{C}^2, \mathbb{Z}_2)$$

which commutes with the boundary map. Moreover

$$\mathbf{M}(\Pi_{\#}(I')) \leq \operatorname{lip} \Pi \mathbf{M}(I') .$$

So as spt $\mathbf{S}_u \subset U$, it is easy to see that $I := \Pi_{\#}(I')$ satisfies the conditions of Proposition 3.1. □

Now we present another important result concerning the maps in $R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$. The same singularity removing proposition was proved in [B2] for $H^1(B^3, S^2)$.

Proposition 3.2. *Let $I \in \mathcal{P}_1(\mathcal{C}^2, \pi_1(\mathbb{RP}^2))$ be a connection for $u \in R^{\infty,1}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$. Then there are maps $v_m \in C^{\infty}_{\varphi}(\mathcal{C}^2, \mathbb{RP}^2)$ such that*

$$\begin{cases} v_m = u \text{ on } \mathcal{C}^2 \backslash K_m \,, \\ |K_m| \leq \frac{1}{m} \,, \\ \int_{\mathcal{C}^2} |\nabla v_m| \leq \int_{\mathcal{C}^2} |\nabla u| + C\mathbf{M}(I) + \frac{1}{m} \cdot \end{cases} \tag{3.13}$$

for some constant $C > 0$ independent of u. □

This proposition is a special case of Proposition 5.1 which is proved in the next section.

3.3 Topological singularities for maps in $W^{1,1+s}(\mathcal{C}^2, \mathbb{RP}^2)$. We give a proof for Theorem II for $M = \mathcal{C}^2$, $N = \mathbb{RP}^2$ and $[p] = 1$. Let u be a map in $W^{1,p}(\mathcal{C}^2, \mathbb{RP}^2)$ such that $[p] = 1$. We intend to define $\mathbf{S}_u$, the topological singular chain of u as a flat $\mathbb{Z}_2$-chain. In fact we are to prove that for any sequence of maps $u_m \in R^{\infty,p}(\mathcal{C}^2, \mathbb{RP}^2) \subset R^{\infty,1}(\mathcal{C}^2, \mathbb{RP}^2)$, $\mathbf{S}_{u_m}$ is a convergent sequence in $\mathcal{F}_0(\mathcal{C}^2, \mathbb{Z}_2)$ and that the limit is independent of the choice of the sequence u_m.

Let u_m be such a sequence. Set as in (3.8)

$$J_m := \int_{\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon} \int_{\mathcal{C}^2} \big| \nabla (p_w \circ u)(x) - \nabla (p_w \circ u_m)(x) \big| dx \, dw \, .$$

We are to prove that $J_m \to 0$. First observe that for fixed $x \in \mathcal{C}^2$

$$\begin{aligned} &\big| \nabla (p_w \circ u)(x) - \nabla (p_w \circ u_m)(x) \big| \\ &\qquad \leq C\big(|\nabla p_w(u(x))| + |\nabla p_w(u_m(x))| \big) \in L^1(\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon) \end{aligned}$$

(see 3.6). Now, since $\nabla(p_w \circ u_m)$ converge for almost every $w \in \mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon$ to $\nabla(p_w \circ u)$, by Lebesgue dominant convergence we get

$$\int_{\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon} \big|\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)\big| dw \to 0$$

for almost every $x \in \mathcal{C}^2$. Also we have

$$\int_{\mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon} \big|\nabla(p_w \circ u)(x) - \nabla(p_w \circ u_m)(x)\big| dw \le C_0(\varepsilon)\big(|\nabla u(x)| + |\nabla u_m|\big) \in L^1(\mathcal{C}^2)\,.$$

Thus, again using the Lebesgue dominant convergence, we obtain that J_m tends to 0 for $m \to +\infty$. As a result, there exists $w \in \mathbb{RP}^2 \backslash \mathcal{G}_\varepsilon$ such that

$$p_w \circ u_m \to p_w \circ u \quad \text{in} \quad W^{1,1}(\mathcal{C}^2, S^1)$$

and that w is a regular value for all u_m, i.e.

$$p_w \circ u_m \in \mathcal{R}^{\infty,1}(\mathcal{C}^2, S^1)\,.$$

Meanwhile, any flat chain with multiplicity in $\mathbb{Z}$ is also a real current, defining a dual functional on the space of compactly supported smooth differential forms. Now if we set $\mathbf{S}_{p_w \circ u}$ to be the real 0-current (distribution) defined as follows:

$$\mathbf{S}_{p_w \circ u}(\alpha) := \frac{1}{2\pi} \int_{\mathcal{C}^2} (p_w \circ u)^*(d\theta) \wedge d\alpha \quad \forall \alpha \in C_c^\infty(\mathcal{C}^2, \mathbb{R})\,,$$

we get

$$m_r(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \to 0$$

where by $m_r(\mathbf{S})$ we mean the minimal mass of normal currents getting $\mathbf{S}$ as their boundary (see [GMS, vol. II, sect. 5.4.2, Theorem 2]). Moreover, for a 0-dimensional integral flat chain $\mathbf{S}$ in $\mathbb{R}^n$ the minimal i.m. rectifiable current taking $\mathbf{S}$ as the boundary is also the minimal real current, i.e. we have

$$m_r(\mathbf{S}) = m_i(\mathbf{S}) := \inf\big\{\mathbf{M}(\mathbf{T}); \mathbf{T} \in \mathcal{R}_1(\mathbb{R}^n) \quad \partial\mathbf{T} = \mathbf{S}\big\}$$

(see [F2]). As a result, $\mathbf{S}_{p_w \circ u}$ is an integral flat ($\mathbf{S}_{p_w \circ u} \in \mathcal{F}_0(\mathcal{C}^2, \mathbb{Z})$) and we get

$$\mathcal{F}(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \le m_i(\mathbf{S}_{p_w \circ u_m} - \mathbf{S}_{p_w \circ u}) \to 0\,.$$

Using Lemma 2.2 and (3.12) we obtain that the flat $\mathbb{Z}_2$-chain

$$\mathbf{S}_u := \chi_*(\mathbf{S}_{p_w \circ u}) = \lim_{m \to \infty} \chi_*(\mathbf{S}_{p_w \circ u_m}) = \lim_{m \to \infty} \mathbf{S}_{u_m}$$

is independent of the choice of w and that $\mathcal{F}(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$. Since any two sequences converging to u can be restructured to a single converging sequence, $\mathbf{S}_u$ is independent of the converging sequence u_m too.

Now suppose that $\mathbf{S}_u = 0$. Consequently for any sequence of maps u_m converging to u in $W^{1,p}(\mathcal{C}^2, S^1)$, there is polyhedral $\mathbb{Z}_2$-chains I_m such that

$$\mathbf{M}(I_m) \to 0$$

and that spt $(\partial I_m - \mathbf{S}_{u_m}) \subset \partial\mathcal{C}^2$ (This is what we call a connection when we do not fix a boundary data.) Using the same method as for the singularity removing Proposition 3.2, we prove the existence of a sequence of smooth maps $v_m : \mathcal{C}^2 \to \mathbb{RP}^2$ which converge to u in $W^{1,1}$. (Here we use the fact that $\mathbf{M}(I_m) \to 0$.) Consequently, u is homotopic to constant on any generic 1-skeleton of $\mathcal{C}^2$. Using this and referring to [B1, proof of Theorem 1], we can approximate strongly u by smooth maps in $W^{1,p}(\mathcal{C}^2, \mathbb{RP}^2)$. This completes the proof of Theorem II for this special case. □

3.4 Study of sequential weak density in $W^{1,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$. We prove Theorem IV for $n = 2$ and $N = \mathbb{RP}^2$: For every $u \in W^{1,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$, there are $u_m \in C^\infty_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$ such that $u_m \to u$ in $L^1(\mathcal{C}^2)$ and ∇u_m converge in the biting sense to ∇u.

Proof. First we approximate u by a sequence $u_k \in R^{\infty,1}_\varphi(\mathcal{C}^2, \mathbb{RP}^2)$ (see Remark 2.1). Passing to a subsequence if necessary, we can assume that energies of u_k are bounded by the same constant. So, by Proposition 3.1, there are polyhedral connections I_k for u_k such that their masses are equibounded. Using Proposition 3.2, we construct maps $u_{k,m}$, which converge almost everywhere to u_k and have equibounded energies too. As a result, $u_{m,m}$ tend in L^1 to u and their gradients are equibounded in L^1 norm. By ([GMS, Vol. I, sect. 1.2.7]), $\nabla u_{m,m}$ converge in L^1 in the biting sense. Furthermore, the limit cannot be other than ∇u, since $u_{m,m}$ converge strongly to u in L^1. □

4 Controlling the Mass of Connections

We assume that $p > 1$ and that N is a $(p-1)$-connected smooth compact manifold of dimension $k \geq p$, i.e.

$$\pi_q(N) = 0 \text{ for } q < p\,.$$

Using the fact that N is $(p-1)$-connected, we generalize the result of Proposition 3.1 to maps in $\mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N)$. This is what we prove in Proposition 4.1. As before, the main idea is to conjugate u with a projection of N on the generators of its p-homotopy group.

Consider some triangulation of N and for $1 \leq l \leq k$, let N^l be the l-skeleton of N. So $N = N^k$. Observe that by ([Wh, Theorem (1.6), p. 215]), N^p is $(p-1)$-connected and the homomorphisms

$$\chi^{p,l} : \pi_p(N^p) \to \pi_p(N^l)\,,$$

induced by the injection maps $i_{p,l} : N^p \to N^l$, are onto. As a result, using [GrM, Corollary 3.5, p. 38], N^p is of the homotopy type of a bouquet of p-spheres and we obtain that $\pi_p(N^p)$ is finitely generated. Let $g_1, \dots, g_\beta$ be its generators. As a result, $\pi_p(N^l)$ is finitely generated too. We choose its generators among $\{\chi^{p,l}(g_1), \dots, \chi^{p,l}(g_\beta)\}$ and we define a norm on $\pi_p(N^l)$, $p \le l \le k$, as follows: For $a \in \pi_p(N^l)$, $|a|$ is the smallest length of a product of generators of $\pi_p(N^l)$ representing a. Observe that there is some constant $C > 0$ such that

$$|\chi^{p,l}(g)| \le C|g| , \quad \forall g \in \pi_p(N^p) . \tag{4.1}$$

Since $\pi_1(N) = 0$, $\mathbf{S}_u \in \mathcal{P}_{n-p-1}(\mathcal{C}^n, \pi_p(N))$ is well defined for any $u \in \mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N)$ (see Definition 2.8). We proceed as before by generalizing the concept of connections:

Definition 4.1. *We say that $\mathbf{T} \in \mathcal{F}_{n-p}(\mathcal{C}^n, \pi_p(N))$ is a connection for $u \in \mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N)$ if $\partial\mathbf{T} = \mathbf{S}_u$.*

We write

$$N^l = \bigcup_{i=1}^{s_l} \xi^l_i(\mathbf{B}^l) ,$$

where

$$\xi^l_i : \mathbf{B}^l \to N^l_i := \xi^l_i(\mathbf{B}^l) , \quad i = 1, \dots, s_l ,$$

are diffeomorphisms and each two N^l_i are rather disjoint or intersecting on a lower dimensional face in N^{l-1}.

Now let $w \in N^l_1 \times \cdots \times N^l_{s_l}$, $w = (w_1, \dots, w_{s_l})$ be such that $w_i \notin N^{l-1}$. Define

$$p^l_w : N^l \backslash \{w_1, \dots, w_{s_l}\} \to N^{l-1}$$

as follows:

$$p^l_w(y) := \begin{cases} \xi^l_i\big(p((\xi^l_i)^{-1}(w_i), (\xi^l_i)^{-1}(y))\big) & \text{if } y \in N^l_i \backslash N^{l-1} , \\ y & \text{otherwise} , \end{cases} \tag{4.2}$$

where p is the projection defined in definition 2.9.

Lemma 4.1. *Let $p + 1 \le l \le k$, then*

(i) p^l_w is well defined and locally lipschitz on $N \backslash \{w_1, \dots, w_{s_l}\}$.

(ii) For any p-dimensional cycle $\mathcal{G}' \subset N \backslash \{w_1, \dots, w_{s_l}\}$ we have

$$[\mathcal{G}']_{\pi_p(N^l)} = \chi^l [p^l_w(\mathcal{G}')]_{\pi_p(N^{l-1})} \tag{4.3}$$

where

$$\chi^l : \pi_p(N^{l-1}) \to \pi_p(N^l)$$

is the homomorphism induced by the injection map $i_l : N^{l-1} \to N^l$.

(iii) *For any* $w' \in N^l$,

$$\int_{N^l_{1,\varepsilon}\times\cdots\times N^l_{s_l,\varepsilon}} \big|\nabla p_w(w')\big|^p dw \le C(p,l,\varepsilon) < +\infty\,, \tag{4.4}$$

where for $1 \le i \le s_l$ *and* $0 < \varepsilon < 1$,

$$N^l_{i,\varepsilon} := \xi^l_i\big(B^l(0,1-\varepsilon)\big)\,. \qquad \square$$

REMARK 4.1. Since N is $(p-1)$-connected, $\pi_p(N) \equiv H_p(N,\mathbb{Z})$ (Hurewicz theorem). So the homotopy class of p-cycles in N is well defined.

Proof. Using (2.5), the lemma is proved as for Lemma 3.1. $\square$

Now let us estimate the integral

$$J := \int_{N^l_{1,\varepsilon}\times\cdots\times N^l_{s_l,\varepsilon}} \int_{\mathcal{C}^n} \big|\nabla(p_w\circ u)(x)\big|^p dx\, dw\,. \tag{4.5}$$

for $u \in W^{1,p}(\mathcal{C}^n, N^l)$, for $p < l$. By (4.4) we have

$$\begin{aligned} J &\le \int_{\mathcal{C}^n}\int_{N^l_{1,\varepsilon}\times\cdots\times N^l_{s_l,\varepsilon}} \big|\nabla p_w(u(x))\big|^p \big|\nabla u(x)\big|^p dw\, dx \\ &\le C(p,l,\varepsilon)\int_{\mathcal{C}^n} |\nabla u|^p. \end{aligned}$$

As a result, by considering (4.5), there is some positive measure set $W \subset N^l_\varepsilon := N^l_{1,\varepsilon}\times\cdots\times N^l_{s_l,\varepsilon} \subset \mathbb{R}^{ls_l}$ for which

$$\int_{\mathcal{C}^n} \big|\nabla(p_w\circ u)\big|^p \le \frac{C(p,l,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} \int_{\mathcal{C}^n} |\nabla u|^p\,, \quad \forall w \in W\,. \tag{4.6}$$

LEMMA 4.2. *Let* $l > p$ *and* $u^l \in \mathcal{R}^{p,\infty}_{w_0}(\mathcal{C}^n, N^l)$ *for some* $w_0 \in N^{l-1}$. *Then there is a map* $u^{l-1} : \mathcal{C}^n \to N^{l-1}$ *and* $C > 0$, *independent of* u^l, *such that*

(i) $u^{l-1} \in \mathcal{R}^{\infty,p}_{w_0}(\mathcal{C}^n, N^{l-1})$;

(ii) $\int_{\mathcal{C}^n} |\nabla u^{l-1}|^p \le C \int_{\mathcal{C}^n} |\nabla u^l|^p$;

(iii) $\mathbf{S}_{u^l} = \chi^l_*(\mathbf{S}_{u^{l-1}})$;

where $\chi^l : \pi_p(N^{l-1}) \to \pi_p(N^l)$ *is the homomorphism induced by the injection map* $i_l : N^{l-1} \to N^l$. $\square$

Proof. Let us fix $0 < \varepsilon < 1$ and consider the set $W \subset N^l_\varepsilon$ as in (4.6). Also we fix ε_1, ε_2, $\varepsilon_3 > 0$ and $0 < \delta < \delta_1$ such that

$$\frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)}\left(K^2\int_{V^{\delta_1}} |\nabla u^l|^p + \delta K\varepsilon_2 + \varepsilon_1\right) + \varepsilon_3 \le \int_{\mathcal{C}^n} |\nabla u^l|^p\,, \tag{4.7}$$

where K, δ and δ_1 satisfy (2.3). For almost all $w = (w_1,\dots,w_{s_l}) \in W$, w_i's are regular values for $u^l|_{\mathcal{C}^n\setminus V^\delta}$ and $u^l|_{\partial V^\delta}$, which are smooth on their domains. Using (2.3) and by the co-area formula we obtain that for almost all $w \in W$, $(u^l)^{-1}(w_i) \cap (\mathcal{C}^n \backslash V^\delta)$ is a finite mass smooth submanifold of $\mathcal{C}^n \backslash V^\delta$, of dimension $n-l$, while its boundary is also a finite mass submanifold of ∂V^δ, of dimension $n-l-1$. We fix such w and we observe that for all $\varepsilon' > 0$, there is $f_{\varepsilon'}$, some lipschitz diffeomorphism of $\mathcal{C}^n$, such that $f_{\varepsilon'}$ is the identity map except on a small neighborhood of $\bigcup_{i=1}^{s_l}(u^l)^{-1}(w_i)$, and we have

$$\begin{cases} f_{\varepsilon'}(V^\delta) = V^\delta\,, \quad f_{\varepsilon'}(\partial V^\delta) = \partial V^\delta, \\ (u^l \circ f_{\varepsilon'})^{-1}(w_i) \cap (\mathcal{C}^n \backslash V^\delta) \text{ is a polyhedral } (n-l)\text{-submanifold of } \mathcal{C}^n \backslash V^\delta \\ (u^l \circ f_{\varepsilon'})^{-1}(w_i) \cap (\partial V^\delta) \text{ is a polyhedral } (n-l-1)\text{-submanifold of } \partial V^\delta, \\ \int_{\mathcal{C}^n} \left|\nabla(u^l \circ f_{\varepsilon'}) - \nabla u^l\right|^p < \varepsilon', \\ \int_{\partial V^\delta} \left|\nabla(u^l \circ f_{\varepsilon'}) - \nabla u^l\right|^p < \varepsilon', \end{cases} \tag{4.8}$$

Let $\varepsilon' = \min\{\varepsilon_1, \varepsilon_2\}$ and denote $v^l := (u^l \circ f_{\varepsilon'})_\delta$. Using (2.3) and (4.8) we get

$$\begin{aligned} \int_{\mathcal{C}^n} |\nabla v^l|^p &= \int_{V^\delta} \left|\nabla(u^l \circ f_{\varepsilon'})_\delta\right|^p + \int_{\mathcal{C}^n \backslash V^\delta} \left|\nabla(u^l \circ f_{\varepsilon'})\right|^p \\ &\le \delta K \int_{\partial V^\delta} \left|\nabla(u^l \circ f_{\varepsilon'})\right|^p + \int_{\mathcal{C}^n \backslash V^\delta} |\nabla u^l|^p + \varepsilon_1 \\ &\le \delta K \varepsilon_2 + \delta K \int_{\partial V^\delta} |\nabla u^l|^p + \int_{\mathcal{C}^n \backslash V^\delta} |\nabla u^l|^p + \varepsilon_1 \\ &\le \int_{\mathcal{C}^n} |\nabla u^l|^p + \left(\delta K \varepsilon_2 + \varepsilon_1 + K^2 \int_{V^{\delta_1}} |\nabla u^l|^p\right). \end{aligned} \tag{4.9}$$

We observe that v^l is continuous on $\mathcal{C}^n \backslash B$ and since $f_{\varepsilon'}$ is a diffeomorphism, it has the same homotopic singularity as u^l on components of B. Now by (4.6) we have

$$\int_{\mathcal{C}^n} \left|\nabla(p_w \circ v^l)\right|^p \le \frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} \int_{\mathcal{C}^n} |\nabla v^l|^p\,. \tag{4.10}$$

So as a result $v^{l-1} := p_w \circ v^l \in W^{1,p}_{w0}(\mathcal{C}^n, N^{l-1})$. Observe that by construction v^{l-1} is locally lipschitz away from

$$\Sigma(v^{l-1}) = \bigcup_{i=1}^{s_l} (u^l \circ f_{\varepsilon'})_\delta^{-1}(w_i) \cup B\,.$$

Moreover by (4.8), $(u^l \circ f_{\varepsilon'})_\delta^{-1}(w_i)$ is a finite union of $(n-l)$-dimensional polyhedra supported in $\mathcal{C}^n$. Thus, since $n-l \le n-p-1$, we can find some $u^{l-1} \in \mathcal{R}^{\infty,p}_{w0}(\mathcal{C}^n, N^{l-1})$ such that u^{l-1} has the same topological singularities as v^{l-1}, and

$$\int_{\mathcal{C}^n} |\nabla u^{l-1} - \nabla v^{l-1}|^p \leq \varepsilon_3 \,.$$

This fact, combined with (4.7), (4.9) and (4.10) yields:

$$\int_{\mathcal{C}^n} |\nabla u^{l-1}|^p \leq \left(\frac{C(l,p,\varepsilon)}{\mathcal{H}^{ls_l}(N^l_\varepsilon)} + 1 \right) \int_{\mathcal{C}^n} |\nabla u^l|^p \,.$$

We have proved so far parts (i) and (ii) of Lemma 4.2. Part (iii) is a direct consequence of (4.3) and the construction of u^{l-1}, using the same argument as in proof of Proposition 3.1 (see (3.12)). □

LEMMA 4.3. *Let N be a $(p-1)$-connected smooth compact manifold. Let $u \in \mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N^p)$ such that φ is constant. Then there exists polyhedral chain $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N^p))$ such that*

$$\begin{cases} \partial T = \mathbf{S}_u \\ \mathbf{M}(T) \leq C \int_{\mathcal{C}^n} |\nabla u|^p \end{cases} \tag{4.11}$$

for some constant $C > 0$ independent of u. □

Proof. As we observed above, N^p is $(p-1)$-connected too and it is finitely generated. Let $g_1, \dots, g_\beta$ be its generators. By ([GrM, Corollary 3.5, P. 38]), we observe that there are smooth maps $p_i : N^p \to S^p$, $i = 1, \dots, \beta$, such that

$$[p_i(\mathcal{G})]_{\pi_p(S^p)} = \alpha_i([\mathcal{G}]_{\pi_p(N^p)}) \quad \text{for any } p-\text{cycle } \mathcal{G} \subset N^p, \tag{4.12}$$

where, for every $a \in \pi_p(N^p)$,

$$a = \sum_{i=1}^{\beta} \alpha_i(a) g_i$$

is its unique decomposition. Meanwhile, for every $u \in \mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, N^p)$, $p_i \circ u$ is in $\mathcal{R}^{\infty,p}_\varphi(\mathcal{C}^n, S^p)$. Since φ is constant, by [ABL] and the approximation theorem 5.6 in [Fl], there is $\mathbf{T}_i \in \mathcal{P}_{n-p}(\mathcal{C}^n, \mathbb{Z})$ such that

$$\begin{cases} \partial \mathbf{T}_i = \mathbf{S}_{p_i \circ u} \\ \mathbf{M}(\mathbf{T}_i) \leq C_i \int_{\mathcal{C}^n} \left|\nabla u(p_i \circ u)\right|^p \end{cases} \tag{4.13}$$

where $C_i > 0$ is independent of u. (See also [P1] for detailed discussion for S^2).

Now consider the one-to-one group homomorphism $\kappa^i : \mathbb{Z} \to \pi_p(N^p)$, $i = 1, \dots, \beta$, defined by $\kappa^i(n) = n g_i$. Observe that we have

$$\sum_{i=1}^{\beta} \kappa^i(\alpha_i(a)) = a \quad \forall a \in \pi_p(N^p) \,,$$

which combined with (4.12) gives

$$\sum_{i=1}^{\beta} \kappa_*^i(\mathbf{S}_{p_i \circ u}) = \mathbf{S}_u \,.$$

Moreover, κ_*^i satisfies

$$\mathbf{M}(\kappa_*^i(T)) \leq C_i' \mathbf{M}(T) \,,$$

for some constant C_i' independent of T. We set

$$\mathbf{T} := \sum_{i=1}^{\beta} \kappa_*^i(\mathbf{T}_i) \,.$$

So $\mathbf{T}$ is a polyhedral $\pi_p(N^p)$-chain, of dimension $n-p$ and supported in $\mathcal{C}^n$. Using Lemma 2.2 and (4.13) we obtain

$$\partial \mathbf{T} = \sum_{i=1}^{\beta} \kappa_*^i(\mathbf{S}_{p_i \circ u}) = \mathbf{S}_u$$

and

$$\mathbf{M}(T) \leq \sum_{i=1}^{\beta} C_i' \mathbf{M}(\mathbf{T}_i) \leq \sum_{i=1}^{\beta} C_i' C_i \int_{\mathcal{C}^n} \left|\nabla(p_i \circ u)\right|^p .$$

This completes the proof since the p_i are smooth. □

Using the above stated lemmas, we prove the following important result:

Proposition 4.1. *For any integer p, $2 \leq p \leq k$, let N be a k-dimensional $(p-1)$-connected compact smooth manifold. Let $\mathcal{C}^n$ be the unit cube in $\mathbb{R}^n$. Then for $u \in R_\varphi^{\infty,p}(\mathcal{C}^n, N)$, there is $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N))$ such that*

$$\begin{cases} \partial \mathbf{T} = \mathbf{S}_u \\ \mathbf{M}(\mathbf{T}) \leq C \int |\nabla u|^p + C \end{cases} \tag{4.14}$$

for some constant $C > 0$ independent of u. □

Corollary 4.1. *For any $u \in R_\varphi^{\infty,p}(\mathcal{C}^n, N)$, there is a minimal connection $\mathbf{T}_u \in \mathcal{F}_{n-p}(\mathcal{C}^n, \pi_p(N))$ which satisfies*

$$\mathbf{M}(\mathbf{T}_u) \leq C \int |\nabla u|^p + C \,.$$

(See Corollary 3.1). □

Proof of Proposition 4.1. It is sufficient to prove the proposition for $\varphi = w_0 \in N^p$, constant. Using the same method as in the proof of Proposition 3.1, combined with the approximation Theorem 5.6 in [Fl], the proof is generalized for any smooth boundary data.

Write $N^k = N$ and $u^k = u$. Using Lemma 4.2 successively we obtain a map $u^p \in \mathcal{R}^{\infty,p}_{w_0}(\mathcal{C}^n, N^p)$, which satisfies

$$\begin{cases} \int_{\mathcal{C}^n} |\nabla u^p|^p \leq C_1 \int_{\mathcal{C}^n} |\nabla u|^p \\ \chi_*(\mathbf{S}_{u^p}) = \mathbf{S}_u \end{cases} \tag{4.15}$$

where $\chi : \pi_p(N^p) \to \pi_p(N)$ is the natural homomorphism and C_1 is independent of u. We apply Lemma 4.3 to u^p and get some $\mathbf{T}_p \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N^p))$ such that

$$\mathbf{M}(\mathbf{T}_p) \leq C_2 \int_{\mathcal{C}^n} |\nabla u^p|^p$$

and

$$\partial \mathbf{T}_p = \mathbf{S}_{u^p}\,.$$

Combining with (4.15) and applying Lemma 2.2, using (4.1), we observe that $\mathbf{T} := \chi_*(\mathbf{T}_p)$ satisfies (4.14). □

5 Removing the Singularities Using Finite Energy

In the section, we prove that we can remove the singularities of a map $u \in \mathcal{R}^{\infty,p}_{\varphi}(\mathcal{C}^n, N)$ by modifying it along one of its polyhedral connections and using an energy almost proportional to the mass of the connection. The idea first appeared in [B2] for $H^1(B^3, S^2)$. Our proof uses a different approach since the situation is technically more involved. Note that we use the same norm defined for $\pi_p(N)$ as in section 4 and the method may not work for non-equivalent norms. This is the exact statement of what we prove in this section:

Proposition 5.1. *Let $p > 1$ be an integer and let N be a k-dimensional simply connected closed manifold. Assume that $\pi_p(N)$ is finitely generated. If $\mathbf{T} \in \mathcal{P}_{n-p}(\mathcal{C}^n, \pi_p(N))$ is a connection for $u \in \mathcal{R}^{\infty,p}_{\varphi}(\mathcal{C}^n, N)$, there are maps $u_m \in C^{\infty}_{\varphi}(\mathcal{C}^n, N)$ such that*

$$\begin{cases} u_m \xrightarrow{L^p} u \quad \text{as } m \to \infty \\ \int_{\mathcal{C}^n} |\nabla u_m|^p \leq \int_{\mathcal{C}^n} |\nabla u|^p + C\mathbf{M}(\mathbf{T}) + O\left(\frac{1}{m}\right) \end{cases} \tag{5.1}$$

for $C > 0$ independent of u. The same result holds when $p = 1$ if $\pi_1(N)$ is abelian. □

First we prove two lemmas necessary for the proof of this proposition.

Lemma 5.1. *For every $g \in \pi_p(N)$, there exists an open covering of N, $\{U_1^g, \dots, U_{\nu_g}^g\}$, and smooth maps*

$$\omega_{g,j} : \mathbf{B}^p \times U_j^g \to N\,, \quad j = 1, \dots, \nu_g\,,$$

such that

$$\begin{cases} \omega_{g,j}(\,.\,|_{\partial \mathbf{B}^p}, y) \equiv y & \forall y \in N \\ \left[\omega_{g,j}(\,.\,,y)\right]_{\pi_p(N)} = g & \forall y \in N \\ \int_{\mathbf{B}^p} \left|\nabla_x \omega_{g,j}(\,.\,,y)\right|^p dx \le C|g| & \forall y \in N \\ |\nabla \omega_{g,j}|_\infty \le C_g \end{cases} \tag{5.2}$$

where $C > 0$ is independent of g and j. □

Proof. Let $h_1, \dots, h_\gamma$ be the generators of $\pi_p(N)$. Since N is compact we can find a finite open covering of N, $\{U_1, \dots, U_\nu\}$, and smooth maps

$$\omega_{i,j} : \mathbf{B}^p \times U_j \to N$$

such that for all i, j and all $y \in N$ we have

$$\begin{cases} \omega_{i,j}(.|_{\partial \mathbf{B}^p}, y) \equiv y \\ \left[\omega_{i,j}(\,.\,,y)\right]_{\pi_p(N)} = h_i\,. \end{cases} \tag{5.3}$$

Now we write $g \in \pi_p(N)$ in its minimal length decomposition

$$g = h_{i_1} + \cdots + h_{i_s}\,,$$

where $s = |g|$. For $y \in N$, $x \in \mathbf{B}^p$ and $\rho = 1, \dots, s$, we set

$$\omega_{g,x}(y) := \omega_{i_\rho, j_\rho}\left(sx - (\rho - 1)\frac{x}{|x|}, y_\rho\right) \quad \text{if} \quad \frac{\rho - 1}{s} \le |x| \le \frac{\rho}{s}\,,$$

where $y_s := y \in U_{j_s}$ and for $\rho = 1, \dots, s-1$,

$$y_\rho := \omega_{i_{\rho+1}, j_{\rho+1}}(0, y_{\rho+1}) \in U_{j_\rho}\,.$$

Observe that by slightly modifying $\omega_{g,x} : \mathbf{B}^p \to N$, we can assume that it is smooth on its domain. Moreover it will satisfy

$$\begin{cases} \omega_{g,y}|_{\partial \mathbf{B}^p} \equiv y \\ [\omega_{g,y}]_{\pi_p(N)} = g \\ \int_{\mathbf{B}^p} |\nabla \omega_{g,y}|^p \le Cs = C|g| \end{cases}$$

for $C > 0$ independent of g and y. Another observation shows that $\omega_{g,y}$ depends smoothly on y in small neighborhoods. Since N is compact, we can find a finite open covering for it, $\{U_1^g, \dots, U_{\nu_g}^g\}$, such that for $j = 1, \dots, \nu_g$

$$\omega_{g,j}(x, y) := \omega_{g,y}(x)\,, \quad \text{if} \quad y \in U_j^g$$

satisfy (5.2). □

Lemma 5.2. *Let $u \in \mathcal{R}_\varphi^{\infty,p}(\mathcal{C}^n, N)$ and $\Sigma \subset \mathcal{C}^n$ be an oriented polyhedral of dimension $n - p$ such that u is continuous on Σ except probably on its boundary. Then for every $g \in \pi_p(N)$, there is a sequence $u_m \in W_\varphi^{1,p}(\mathcal{C}^n, N)$*

and $C>0$ independent of g and u such that

$$\begin{cases} u_m = u \quad \textit{on} \quad \mathcal{C}^n \backslash K_m \\ |K_m| \to 0 \quad \textit{as} \quad m \to \infty \\ \int_{\mathcal{C}^n} |\nabla u_m|^p \le \int_{\mathcal{C}^n} |\nabla u|^p + C|g||\Sigma| + \frac{1}{m} \end{cases} \tag{5.4}$$

and

$$\mathbf{S}_{u_m} = \mathbf{S}_u - g[[\partial \Sigma]] \,. \qquad \square$$

Proof. We identify $\mathbb{R}^n$ with $\mathbb{R}^{n-p} \times \mathbb{R}^p$ with variables $X \in \mathbb{R}^{n-p}, Y \in \mathbb{R}^p$. Without loss of generality we can assume that Σ lies in the plane $\mathbb{R}^{n-p} \times \{0\}$. We divide Σ in polyhedra of equal dimension

$$\Sigma := \bigcup_{j=1}^{\nu_g} \bigcup_{i=1}^{i_j} \Sigma_j^i$$

such that $u(\Sigma_j^i) \subset U_j^g$ for all i,j. We choose B as in section 2.2 such that

$$\bigcup_{j=1}^{\nu_g} \bigcup_{i=1}^{i_j} \partial \Sigma_j^i \subset B$$

and we replace u by u_{δ_1} for δ_1 small enough (see Definition 2.5). This doesn't much change the energy of u and $\mathbf{S}_{u_{\delta_1}} = \mathbf{S}_u$, so it is sufficient to prove the lemma for $u = u_{\delta_1}$. Since u is radial, we have for some constant $C_1 > 0$

$$|\nabla u(x)| \le C_1 \text{ if } x \in \mathcal{C}^n \backslash V_{\delta_1}, \quad |\nabla u(x)| \le \frac{C_1}{\|x - B\|} \text{ if } x \in V_{\delta_1} \,. \tag{5.5}$$

We set for $\eta \ll \delta < \delta_1$ and $(X,Y) \in \mathcal{C}^n \backslash V_\delta$

$$v^{\delta,\eta}(X,Y) := \begin{cases} u(X,Y) & \text{if } (X,0) \notin \Sigma \text{ or if } |Y| \ge \eta \\ u\big(X, 2Y - \eta\frac{Y}{|Y|}\big) & \text{if } (X,0) \in \Sigma \text{ and } \frac{\eta}{2} \le |Y| \le \eta \\ \omega_{g,j}\big(\frac{2}{\eta}Y, u(X,0)\big) & \text{if } (X,0) \in \bigcup_{i=1}^{i_j} \Sigma_j^i \text{ and } |Y| \le \frac{\eta}{2} \,. \end{cases} \tag{5.6}$$

We set

$$\Sigma^\eta := \big\{(X,Y);\, (X,0) \in \Sigma,\, |Y| \le \eta\big\} \,.$$

and we observe that $\mathrm{vol}(\partial V^\delta \cap \Sigma^\eta) = O(\eta^p)$. Using (2.2), (2.3) and (5.5) we get

$$\int_{V_\delta} \big|\nabla(v^{\delta,\eta} \circ h_\delta) - \nabla u\big|^p \le \delta K \int_{\partial V_\delta \cap \Sigma^\eta} \left(\frac{C_g C_2}{\eta} + C_1\right)^p \le O(\delta)$$

for $C_2 > 0$ independent of δ. Moreover for fixed δ we have

$$\begin{aligned} \int_{\mathcal{C}^n \backslash V_\delta} |\nabla v^{\delta,\eta} - \nabla u|^p &\le C|g||\Sigma| + \int_{\Sigma^\eta} \left(\frac{C_g C_2}{\delta} + C_1\right)^p \\ &\le C|g||\Sigma| + O(\eta) \,. \end{aligned}$$

As a result, by choosing successively suitable δ and η, $u_m := (v^{\delta,\eta})_\delta$ will satisfy (5.4). Moreover we have

$$\mathbf{S}_{u_m} = \mathbf{S}_u \pm g[[\partial\Sigma]]\,.$$

If necessary, we get the good sign by replacing g by $-g$ above. □

Proof of Proposition 5.1. We write

$$\mathbf{T} = \sum_{i=1}^{\theta} g_i[[\Sigma^i]]\,.$$

Put $u_m^0 := u$ and for $i = 1, \dots, \theta$, let u_m^i be the m-th element of the sequence obtained by applying Lemma 5.2 to u_m^{i-1} for Σ^i, g_i. We get

$$\begin{aligned} \mathbf{S}_{u_m^\theta} &= \mathbf{S}_u - \sum_{i=1}^{\theta} g_i[[\partial\Sigma^i]] \\ &= \mathbf{S}_u - \partial\mathbf{T} = 0\,, \end{aligned}$$

and we observe that u_m^θ satisfy (5.1). Pay attention that $\mathbf{S}_{u_m^\theta} = 0$ means that u_m^θ, restricted to almost every small enough p-cycle in $\mathcal{C}^n$, is homotopic to constant in N. Using this and referring to [B1, proof of Theorem 1], we can strongly approximate u_m^θ by smooth maps in $C_\varphi^\infty(\mathcal{C}^n, N)$. This completes the proof. □

6 Proof of Theorems II, III and IV

Theorem II is proved using the same arguments as for $W^{1,1+\varepsilon}(\mathcal{C}^2, \mathbb{RP}^2)$, regarding the fact that we have developed the necessary tools above. Observe that the equality

$$m_i(\mathbf{S}) = m_r(\mathbf{S})$$

holds true for any integral flat chain $\mathbf{S}$ in $\mathbb{R}^n$ if and only if $\mathbf{S}$ is of dimension 0 or codimension 2 in $\mathbb{R}^n$ (see [F2]). Thus our method cannot be used for $[p]$ taking a value other than 1 or $n-1$.

Considering Propositions 4.1 and 5.1, Theorem III is proved the same as in section 3.4. The only difference is that since $p > 1$, a bounded sequence in $W_\varphi^{1,p}(\mathcal{C}^n, N)$ has a weakly convergent subsequence.

Propositions 4.1 and 5.1 hold for $p = 1$, abelian $\pi_1(N)$, thus Theorem IV is proved with the same method.

7 Global Singularities and Inverse Images

In this section we will prove Theorem I. That is, we will prove that if M is a smooth simply connected compact manifolds, then smooth maps are sequentially sense in $W^{1,2}(M,N)$ for any smooth closed manifold N.

The idea is essentially the same as in previous sections. We will not repeat some technical details in order to concentrate on the main difference between local and global singularity cases. Also we assume that M is closed. The theorem can be proved by slight modifications as exposed in previous cases for the general situation.

As before, we define $\mathcal{R}^{2,\infty}(M,N)$ to be the space of maps which are smooth except on a finite union of k-dimensional submanifolds of M for $k \leq n-3$ (n=dim$M$). $\mathcal{R}^{2,\infty}(M,N)$ is strongly dense in $W^{1,2}(M,N)$ (see [B1] and [HaL2]), and any map $u \in W^{1,2}(M,N)$ can be strongly approximated by a sequence of maps in $\mathcal{R}^{2,\infty}(M,N)$. Then the theorem is reduced to prove that any map in $\mathcal{R}^{2,\infty}(M,N)$ can be approximated weakly by smooth maps with equibounded energy.

Naturally we can define $\mathbf{S}_u$, the local topological singularity of any map $u \in \mathcal{R}^{2,\infty}(M,N)$ using the homotopy class of u around its $(n-3)$-dimensional singularities. By methods described above, we can approximate u by a sequence of equibounded maps $u_m \in \mathcal{R}^{2,\infty}(M,N)$ for which $\mathbf{S}_{u_m} = 0$. But the condition $\mathbf{S}_{u_m} = 0$ would not be sufficient for approximating this map by smooth maps in the strong topology. We have described this situation in our introduction, regarding an example given by F. Hang and F.H. Lin [HaL1].

We approximate $u \in \mathcal{R}^{2,\infty}(M,N)$ with equibounded maps which satisfy a stronger condition. This condition, introduced in [HaL2] by the authors, is the necessary and sufficient condition for a map in $\mathcal{R}^{2,\infty}(M,N)$ to be strongly approximable by smooth maps: Indeed, if for $u \in \mathcal{R}^{2,\infty}(M,N)$, $u|_{M^2}$ is extendable to a smooth map $\tilde{u} : M \to N$ for every "generic" 2-skeleton M^2 of M, then u can be approximated by smooth maps in $W^{1,2}(M,N)$ (see [HaL2, Theorem 6.2]). So for proving the theorem it suffices to prove the following lemma:

Definition 7.1. *$u \in \mathcal{R}^{2,\infty}(M,N)$ satisfies the 2-skeleton condition if and only if $u|_{M^2}$ is extendable to a smooth map $\tilde{u} : M \to N$ for every "generic" 2-skeleton M^2 of M.*

Lemma 7.1. *Any $u \in \mathcal{R}^{2,\infty}(M,N)$ can be approximated in L^2 by a sequence of maps $u_m \in \mathcal{R}^{2,\infty}(M,N)$ such that the u_m satisfy the 2-skeleton* *condition and that for $C > 0$ independent of u we have*

$$\int_M |\nabla u_m|^2 \le C \int_M |\nabla u|^2. \qquad \square$$

Proof. Let us fix some notation. Let N^2 be the 2-skeleton of N for some cubization. Also let $\mathcal{P} : N \to N^2$ be the projection obtained by mapping inductively N^l on N^{l-1} by the method described in section 4, i.e.

$$\mathcal{P} := p^3_{w^3} \circ \cdots \circ p^k_{w^k}$$

where $p^l_{w^l}$ is the map defined in (4.2). By choosing suitable w^l we can be sure that for some constant $C > 0$ independent of u we have

$$\int_M |\nabla u^2|^2 \le C \int_M |\nabla u|^2 \,,$$

where $u^2 := \mathcal{P} \circ u$. We consider also the onto group homomorphism $\chi^{2,k}$: $\pi_2(N^2) \to \pi_2(N)$ which we will refer to simply by χ. So as in Lemma 4.1 we have

$$\chi\big([\mathcal{P}(\mathcal{G})]_{\pi_2(N^2)}\big) = [\mathcal{G}]_{\pi_2(N)} \,, \tag{7.1}$$

for any 2-cycle $\mathcal{G}$ in N.

Now as in Lemma 4.3 we observe that N^2 is simply connected too and that $\pi_2(N^2)$ is finitely generated. Let $g_1, \dots, g_\beta$ be its generators. By [GrM, Corollary 3.5, P. 38], we observe that there are smooth maps $p_i : N^2 \to S^2$, $i = 1, \dots, \beta$, such that

$$[p_i(\mathcal{G})]_{\pi_2(S^2)} = \alpha_i([\mathcal{G}]_{\pi_2(N^2)}) \quad \text{for any } 2-\text{cycle } \mathcal{G} \subset N^2, \tag{7.2}$$

where, for every $a \in \pi_2(N^2)$,

$$a = \sum_{i=1}^{\beta} \alpha_i(a) g_i$$

is its unique decomposition. Meanwhile, for every $u \in \mathcal{R}^{\infty,2}(M, N^2)$, $p_i \circ u$ is in $\mathcal{R}^{\infty,2}(M, S^2)$. By using the co-area formula as in [ABL] we obtain that there are points $y_i \in S^2$, $i = 1, \dots, \beta$, for which the inverse image $T_i := (p_i \circ u^2)^{-1}(y_i)$ is a smooth $(n-2)$-dimensional submanifold of M and that

$$\mathcal{H}^{n-2}(T_i) \le C \int_M |\nabla u^2|^2$$

for some constant $C > 0$ independent of the choice of u. We orient the T_i using the standard orientation frame on S^2.

We now construct the maps u_m which satisfy the conditions of the lemma. Using the same methods as in Lemma 5.2, we modify u along the inverse images T_i in their δ-neighborhoods by introducing in 2-dimensional topological disks orthogonal to T_i in M new maps generating $\chi(g_i)$. We can do this by using a controlled amount of energy since the volume of each T_i is controlled by the energy of u and the energy necessary for realizing the $\chi(g_i)$, $i = 1, \dots, \beta$, as a cycle, based on any point of N, is uniformly bounded (see Lemma 5.1). So by tending δ to zero we obtain a sequence of maps $u_m \in \mathcal{R}^{2,\infty}(M, N)$, converging in L^2 to u. We should prove that the u_m satisfy the 2-skeleton condition to prove the lemma.

By (7.1) and (7.2), we obtain, for any generic 2-cycle $\mathcal{G}$ in M,

$$\big[p_i \circ \mathcal{P} \circ u_m(\mathcal{G})\big]_{\pi_2(S^2)} = \big[p_i \circ u^2(\mathcal{G})\big]_{\pi_2(S^2)} \pm \sum_{j=1}^{\beta} n_j \alpha_i(k_j)\,,$$

where n_j is the topological intersection number between $\mathcal{G}$ and T_j in M and $\chi(k_j) = \chi(g_j)$. Observe that as u_m is in $\mathcal{R}^{2,\infty}(M, N)$, it is smooth on any generic 2-cycle $\mathcal{G} \subset M$ or on any generic 2-skeleton M^2 of M. Meanwhile another simple topological observation shows that

$$\big[p_i \circ u^2(\mathcal{G})\big]_{\pi_2(S^2)} = n_i\,.$$

Combining these facts with (7.1) and (7.2), and by changing, if necessary, the orientation of our modifying maps in the transversal disks, we obtain, for any generic 2-cycle in M,

$$\begin{aligned}
\big[u_m(\mathcal{G})\big]_{\pi_2(N)} &= \chi\bigg(\sum_{i=1}^{\beta} \big[p_i \circ \mathcal{P} \circ u_m(\mathcal{G})\big]_{\pi_2(S^2)} g_i\bigg) \\
&= \sum_{i=1}^{\beta} \bigg(n_i - \sum_{j=1}^{\beta} n_j \alpha_i(k_j)\bigg)\chi(g_i) \\
&= \sum_{i=1}^{\beta} n_i \chi(k_i) - \sum_{i=1}^{\beta}\sum_{j=1}^{\beta} n_j \alpha_i(k_j)\chi(g_i) \\
&= \sum_{i=1}^{\beta} n_i \sum_{j=1}^{\beta} \alpha_j(k_i)\chi(g_j) - \sum_{i=1}^{\beta}\sum_{j=1}^{\beta} n_j \alpha_i(k_j)\chi(g_i) = 0\,.
\end{aligned}$$

As a result, for any generic 2-skeleton M^2 of M, $(u_m|_{M^2})_* : \pi_2(M^2) \to \pi_2(N)$, the induced group homomorphism of u on the second homotopy groups of M^2 and N is the trivial map. Meanwhile $\pi_1(M) = 0$, so M^2 is of the same homotopy type of a bouquet of spheres. Consequently, $u_m|_{M^2}$ is homotopic to a constant in N. So it can be extended to a smooth map from M into N. The u_m satisfy the 2-skeleton condition: This completes the proof of the lemma and consequently that of the Theorem I for the case $\pi_1(N) = 0$.

Now let N be any closed manifold for which $\pi_1(N) \neq 0$. Trying to adapt the method used for proving the theorem for the simply connected case, the main problem to overcome is that in this case N^2 may not be of the same homotopy type as a bouquet of spheres. Another problem is that $\pi_2(N^2)$ may be infinitely generated, which gives us some problems in controlling the energy of the modified maps.

For overcoming these problems we should use an approach different from that of CW-complexes. That is, in place of a cubization of N, we will consider this time its handle decomposition in the increasing index order. The readers can refer to [GoS] for detailed information on handles. Briefly, a handle of index i is a copy of $D^i \times D^{k-i}$, attached to the boundary of a k-manifold X along $\partial D^i \times D^{k-i}$ by an embedding $\varphi : \partial D^i \times D^{k-i} \to \partial X$. Note that attaching an i-handle is equivalent to attaching an i-cell up to homotopy as we can think of an i-handle as an i-cell thickened up to be k-dimensional. We refer to this i-cell as the core of the handle. By a handle decomposition of N we mean an identification of N with a smooth manifold obtained from the empty set by attaching handles. There is always a handle decomposition of the closed smooth manifold N so that the handles are attached in order of increasing index, where the handles of the same index can be attached in any order (see [GoS, Proposition 4.2.7]). From now on, by N^l, we refer to the smooth k-manifold (probably with boundary) obtained after attaching the l-handles in the handle decomposition of N. The N^l will play the same role here as the l-skeleton of N in the previous parts of this paper, which we had similarly referred to by N^l.

We consider the smooth Riemannian manifold $\widetilde{N}$, the universal covering of N, and the corresponding fibration $F : \widetilde{N} \to N$. We assume that $\widetilde{N}$ is equipped with the pullback metric under F and is embedded isometricly in some $\mathbb{R}^{N'}$ such that F is a local isometry. We consider N^2 as defined in the last paragraph and again we observe that $\pi_1(N) = \pi_1(N^2)$ and for $2 \leq l \leq k$, the homomorphisms

$$\chi^{2,l} : \pi_2(N^2) \to \pi_2(N^l)\,,$$

induced by the injection maps $i_{2,l} : N^2 \to N^l$, are onto (see [GoS, p. 111]). Set

$$\widetilde{N^2} := F^{-1}(N^2)\,.$$

Since $\pi_1(N^2) = \pi_1(N)$ and using the homotopy theory, we deduce that $\widetilde{N^2}$ is the universal covering of N^2 as a smooth k-manifold and that $F|_{\widetilde{N^2}}$ is the corresponding fibration. Observe that this diagram is commutative:

$$\begin{array}{ccc}\pi_2(\widetilde{N^2}) & \xrightarrow{\widetilde{\chi}^{2,k}} & \pi_2(N)\\ \Big\downarrow (F|_{\widetilde{N^2}})_* & & \Big\downarrow F_*\\ \pi_2(N^2) & \xrightarrow{\chi^{2,k}} & \pi_2(N)\end{array} \tag{7.3}$$

where $\widetilde{\chi}^{2,k} : \widetilde{N^2} \to \widetilde{N}$ is induced by the injection map $\tilde{i}_{2,k} : \widetilde{N^2} \to \widetilde{N}$ and is onto. Also

$$F_* : \pi_2(\widetilde{N}) \to \pi_2(N) \quad \text{and} \quad (F|_{\widetilde{N^2}})_* : \pi_2(\widetilde{N^2}) \to \pi_2(N^2)$$

are isomorphisms. Thus, since $\pi_1(\widetilde{N^2}) = \pi_1(\widetilde{N}) = 0$, using ([GrM, Corollary 3.5, P. 38]) we obtain that $\widetilde{N^2}$, when deformed smoothly into its 2-dimensional core, is of the homotopy type of a probably infinite bouquet of spheres.

Any $u \in \mathcal{R}^{2,\infty}(M,N)$ can be lifted to a map $\tilde{u} : M \to \widetilde{N}$ as $\pi_1(M\backslash\Sigma(u)) = 0$. (Remember that $\pi_1(M) = 0$ and that $\Sigma(u)$ is of codimension 3 in M). Since F is a local isometry, we get that $\tilde{u} \in \mathcal{R}^{2,\infty}(M,N)$ and that

$$\int_M |\nabla\tilde{u}|^2 = \int_M |\nabla u|^2 . \tag{7.4}$$

Let $u \in W^{1,2}(M,N)$ and $u_m \in \mathcal{R}^{2,\infty}(M,N)$ a sequence converging strongly to u. Using the same method as in Proposition 4.1 we can prove the existence of some constant $C > 0$ independent of u_m, and maps $u_m^2 \in \mathcal{R}^{2,\infty}(M,N^2)$ such that

$$\int_M |\nabla u_m^2|^2 \le C \int_M |\nabla u_m|^2. \tag{7.5}$$

In fact, here we define the projections $p^l : N^l \to N^{l-1}$ using the handle decomposition of N. $p^l : N^l \to N^{l-1}$ is defined by tearing up the l-handles attached to N^{l-1} on some $k-l$ dimensional disk transversal to their cores and projecting the handle on its boundary lying in ∂N^{l-1}. Since the dimension of the core is greater than 2 for $l > 2$, by an average type argument as in the proof of Proposition 4.1, we can be sure that for a suitable choice of the tearing point in the core,

$$\int_M |\nabla p^l \circ u^l|^2 \le C^l \int_M |\nabla u^l|^2$$

when $u^k := u$, $u^l := p^{l+1} \circ u^{l+1}$ are defined by induction and the C^l are independent of the choice of $u \in \mathcal{R}^{2,\infty}(M,N)$.

Now consider the liftings $\tilde{u}_m^2 \in \mathcal{R}^{2,\infty}(M, \widetilde{N^2})$. We observe that $\widetilde{N^2}$ is simply connected. Let $\{g_i\,;\, i \in \mathbb{N}\}$ be the generators of $\pi_2(\widetilde{N^2})$. Recall that $\pi_1(N)$ acts on $\widetilde{N}^2$ and the action is isometric and transitive. We show this action by the function $h(\cdot)$ for $h \in \pi_1(N)$. Then $h_* : \pi_2(\widetilde{N^2}) \to \pi_2(\widetilde{N^2})$ is the induced isomorphism corresponding to the action of h on $\widetilde{N}^2$.

LEMMA 7.2. *There is a finite subset, $\{g_1, \dots, g_s\}$, of the generators of $\pi_2(\widetilde{N^2})$ such that for any generator g_i of $\pi_2(\widetilde{N^2})$, there exists $h \in \pi_1(N)$ and $1 \le j \le s$ such that $g_i = h_*(g_j)$. Moreover there are smooth maps $p_i : N^2 \to S^2$, $i = 1 \in \mathbb{N}$, such that*

$$\big[p_i(\mathcal{G})\big]_{\pi_2(S^2)} = \alpha_i\big([\mathcal{G}]_{\pi_2(\widetilde{N^2})}\big) \quad \text{for any } 2-\text{cycle } \mathcal{G} \subset N^2, \tag{7.6}$$

where, for every $a \in \pi_2(\widetilde{N^2})$,

$$a = \sum_{i=1}^{\infty} \alpha_i(a) g_i$$

is its unique decomposition. Meanwhile, for every $v \in \mathcal{R}^{\infty,2}(M, \widetilde{N^2})$, $p_i \circ v$ is in $\mathcal{R}^{\infty,2}(M, S^2)$. In fact we have

$$\sum_{i=1}^{\infty} \int_M \big|\nabla(p_i \circ v)\big|^2 \le C \int_M |\nabla v|^2 \tag{7.7}$$

for some $C > 0$ independent of v. Also if $g_i = h_*(g_j)$ for $h \in \pi_1(N)$, then $p_j = p_i \circ h$. □

Proof. This lemma is deduced from the fact that $\widetilde{N}^2$ is the universal covering of N^2 which is compact and has a finite handle body decomposition. By retracting the handles onto their cores in increasing index order, we observe that N^2 can be retracted smoothly onto a bouquet of smooth manifolds $\mathcal{N}^2 = \bigvee_{i=1}^s S_i$, where the S_i are either versions of S^1 or compact surfaces. Therefore, $\mathcal{N}^2$ is a basic bouquet of compact surfaces, $\mathcal{N}_0^2$, for which $\pi_2(\mathcal{N}_0^2) = 0$, to which we have attached a finite number of topological S^2's or $\mathbb{RP}^2$'s which we rename to $S_1, \dots, S_s$. Observe that the inverse image of any point in $\mathcal{N}^2$ under the retraction map is simply-connected. Consequently, $\widetilde{N}^2$ can be retracted smoothly onto $\widetilde{\mathcal{N}}^2$, a universal covering for $\mathcal{N}^2$, which contains a union of infinitely many disjoint topological spheres $S_i^2, i \in \mathbb{N}$, relied by a simply-connected skeleton to each other such that the intersection of each S_i^2 with the skeleton is a set of isolated points. As an example if $\mathcal{N}^2$ is $S^1 \vee S^2$, then $\widetilde{\mathcal{N}}^2$ would be a countable number of spheres attached to a line. Observe that there is a natural one-to-one correspondence between the g_i, the generators of $\pi_2(\widetilde{N}^2)$ and the S_i^2. Also for any $i \in \mathbb{N}$, there exists $1 \leq j \leq s$ such that $F(S_i^2) = S_j$, where F is the covering map. Thus if $g_i = h_*(g_j)$ for $h \in \pi_1(N)$, then $S_i^2 = h(S_j^2)$. We define $\hat{p}_i$ to be the identity map on S_i^2 and to map $\widetilde{\mathcal{N}}^2 \backslash S_i^2$ to the corresponding fixed points in S_i^2. Also for $1 \leq j \leq s$, let $\tilde{p}_j : \mathcal{N}^2 \to S_j$ be the natural projections of the bouquet $\mathcal{N}^2$ onto its components. If S_j is a sphere, then $\tilde{p}'_j$ is defined to be the conjugation of $\tilde{p}_j \circ F$ with a fixed diffeomorphism of S_j into S^2. If S_j is a projective space, we should first lift $\tilde{p}_j \circ F$ to the universal covering of $S_j = \mathbb{RP}^2$ which would be a topological sphere. We put

$$p_i := \tilde{p}'_j \circ \hat{p}_i : \widetilde{\mathcal{N}}^2 \to S^2$$

for when $F(S_i^2) = S_j$. Observe that if $g_i = h_*(g_j)$, then $p_j = p_i \circ h$. Since $\widetilde{\mathcal{N}}^2$ is a deformation retract of $\widetilde{N}^2$ which is a lifting of a deformation retract of N^2 onto $\mathcal{N}^2$, we can extend the p_i and the $\tilde{p}'_j$ onto $\widetilde{N}^2$ by conserving the same properties. The lemma is a straightforward consequence of this construction. □

PROPOSITION 7.1. *For $i \in \mathbb{N}$, there is $y_i \in S^2$, the image of a unique point $\hat{y}_i \in S_i^2$ under p_i, and a regular value for any $p_i \circ \tilde{u}_m$, such that if $p_j = p_i \circ h$ for $h \in \pi_1(N)$, then $y_i = y_j$. Moreover for a subsequence of u_m the inverse image $T_i^m := (p_i \circ \tilde{u}_m^2)^{-1}(y_i)$ is a smooth $(n-2)$-dimensional submanifold of M and*

$$\sum_{i=1}^{\infty} \mathcal{H}^{n-2}(T_i^m) \leq C \int_M |\nabla u|^2 + C$$

for $C > 0$ independent of m. □

Proof. By using the co-area formula as in [ABL] we obtain

$$\sum_{j=1}^{s} int_{S^2} \mathcal{H}^{n-2}\big((\tilde{p}'_j \circ \tilde{u}_m^2)^{-1}(y)\big)dy \leq C \int_M |\nabla \tilde{u}_m^2|^2 \leq C \int_M |\nabla u|^2 + C\,.$$

Thus, by Fatou's lemma

$$\int_{S^2} \liminf_{m \to +\infty} \sum_{j=1}^{s} \mathcal{H}^{n-2}\big((\tilde{p}'_j \circ \tilde{u}_m^2)^{-1}(y)\big)dy \leq C \int_M |\nabla u|^2 + C\,.$$

As a result, the subset

$$\left\{ y \in S^2;\ \liminf_{m \to +\infty} \sum_{j=1}^{s} \mathcal{H}^{n-2}\big((\tilde{p}'_j \circ \tilde{u}_m^2)^{-1}(y)\big) \leq \frac{1}{4\pi}\left(C \int_M |\nabla u|^2 + C\right) \right\}$$

is of positive measure in S^2. This, combined with Sard's theorem yields that there are points $y_j \in S^2$ for which the inverse images $\widetilde{T}_j^m := (\tilde{p}'_j \circ \tilde{u}_m^2)^{-1}(y_j)$

are smooth $(n-2)$-dimensional submanifold of M and that

$$\sum_{j=1}^{s} \mathcal{H}^{n-2}(\widetilde{T}_j^m) \leq C \int_M |\nabla u|^2 + C$$

for some constant $C > 0$ independent of the choice of m. For $i \in \mathbb{N}$ let $y_i := y_j$ if $F(S_i^2) = S_j$. Note that $(\tilde{p}_j')^{-1}(y_j)$ is the disjoint union of the $p_i^{-1}(y_i)$ over all the i satisfying $F(S_i^2) = S_j$. Thus the y_i satisfy the conditions of the proposition. □

We orient the T_i^m using the standard orientation frame on S^2. Now observe that $u_m(T_m^i) \subset B_i := (p_i \circ \mathcal{P})^{-1}(y_i)$ which is a smooth compactly supported submanifold of $\widetilde{N}$. We are to construct the maps $\tilde{v}_m$ which will satisfy the 2-skeleton condition with respect to M and $\widetilde{N}$. Using the same methods as in Lemma 5.2, we modify $\tilde{u}_m$ along the inverse images T_i^m in their δ-neighborhoods by introducing in 2-dimensional topological disks orthogonal to T_i^m in M new maps generating $\widetilde{\chi}^{2,k}(g_i)$. We can do this by using a controlled amount of energy since the sum of the volumes of the T_i^m is controlled by the energy of u and the energy necessary for realizing the $\widetilde{\chi}^{2,k}(g_i)$, as a cycle, based on a point in B_i, is uniformly bounded independent of i. In fact, by Lemma 7.2 and Proposition 7.1, for every i, there exists $h \in \pi_1(N)$ and $1 \leq j \leq s$ such that $g_i = h_*(g_j)$ and $B_i = h(B_j)$. However the action of h on $\widetilde{N}$ is isometric, so to control the energy necessary to generate $\widetilde{\chi}^{2,k}(g_i)$ along T_m^i it suffices for us to control the energy necessary to generate $\widetilde{\chi}^{2,k}(g_j)$, for $j = 1, \dots, s$, based on points in B_j. This is possible as the B_j are compactly supported in $\widetilde{N}$. By tending δ to zero we obtain a sequence of equibounded maps $\tilde{v}_m \in \mathcal{R}^{\infty,2}(M, \widetilde{N})$, converging in L^2 to $\tilde{u}$. As in the simply connected case we can prove that $\tilde{v}_m \in \mathcal{R}^{2,\infty}(M, N)$ satisfy the 2-skeleton condition with respect to M and $\widetilde{N}$. As a result, $F \circ \tilde{v}_m \in \mathcal{R}^{2,\infty}(M, N)$ will satisfy the same condition with respect to M and N. This completes the proof of the theorem. □

References

[ABL] F.J. Almgren, W. Browder, E.H. Lieb, Co-area, liquid crystals, and minimal surfaces, in "Partial Differential Equations", Springer Lecture Notes in Math. 1306 (1988), 1–22.

[B1] F. Bethuel, The approximation problem for Sobolev maps between two manifolds, Acta Math. 167 (1992), 153–206.

[B2] F. Bethuel, A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps, Ann. Inst. Henri Poincaré 7 (1990), 269–286.

[BB] F. Bethuel, H. Brezis, Regularity of minimizers of relaxed problems for harmonic maps, J. Funct. Anal. 101 (1991), 145–161.

[BBC] F. Bethuel, H. Brezis, J.-M. Coron, Relaxed energies for harmonic maps, in "Variational Methods" (H. Berestycki, J.-M. Coron, J. Ekeland, eds), Birkhäuser, Basel, 1990.

[BCDH] F. Bethuel, J.-M. Coron, F. Demengel, F. Hélein, A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds, in "Nematics, Mathematical and Physical Aspects", (J.-M. Coron, J.-M. Ghidaglia, F. Hélein), NATO ASI Series C, Kluwer Academic Publishers, Dordrecht 332 (1991), 15–23.

[BZ] F. Bethuel, X. Zheng, Density of smooth functions between two manifolds in Sobolev spaces, J. Funct. Anal. 80 (1988), 60–75.

[BrC] H. Brezis, J.-M. Coron, Large solutions for harmonic maps in two dimensions, Comm. Math. Phys. 92 (1983), 203–215.

[BrCL] H. Brezis, J.-M. Coron, E. Lieb, Harmonic maps with defects, Comm. Math. Phys. 107 (1986), 649–705.

[DH] F. Demengel, R. Hadiji, Relaxed energy for functionals on $W^{1,1}(B^2, S^1)$, Nonlin. Anal. Theo., Meth. and Appl. 19:7 (1992), 625–641.

[F1] H. Federer, Geometric Measure Theory, Springer-Verlag, Berlin, 1969.

[F2] H. Federer, Real flat chains, cochains and variational problems, Ind. Univ. Math. J. 24 (1974), 351–407.

[Fl] W. Fleming, Flat chains over a coefficient group, Trans. Amer. Math. Soc. 121 (1966), 160–186.

[GMS] M. Giaquinta, G. Modica, J. Soucek, Cartesian Currents in the Calculus of Variations, Springer-Verlag, Berlin, 1998.

[GoS] R.E. Gompf, A.I. Stipsicz, 4-Manifolds and Kirby Calculus, Graduate Studies in Mathematics 20, Amer. Math. Soc., 1999.

[GrM] P.A. Griffith, J.W. Morgan, Rational Homotopy Theory and Differential Forms, Birkhäuser, Boston, 1981.

[Gro] M. Gromov, Structure métrique pour les variétés riemanniennes, Cedic/Fernand Nathan, Paris, 1981.

[H] P. Hajlasz, Approximation of sobolev mappings, Nonlin. Anal., Theory, Methods and Applications 22:12 (1994), 1579–1591.

[Ha] F. Hang, Density problems for $W^{1,1}(M, N)$, Comm. Pure Appl. Math. 55:7 (2002), 937–947.

[HaL1] F. Hang, F.H. Lin, Topology of Sobolev mappings, Math. Res. 8:3 (2001), 321–330.

[HaL2] F. Hang, F.H. Lin, Topology of Sobolev mappings II, preprint (2001).

[HarR] R. Hardt, T. Rivière, Connecting Topological Hopf Singularities, preprint (2001).

[Hu] D. Husemoller, Fibre Bundles, second ed., Springer-Verlag, New York, 1975.

[I] T. Isobe, Energy gap phenomenon and the existence of infinitely many weakly harmonic maps for the Dirichlet problem, J. Funct. Anal. 129 (1995), 243–267.

[P1] M.R. Pakzad, Existence of infinitely many weakly harmonic maps from a domain in $\mathbb{R}^n$ into S^2 for non-constant boundary data, Cal. Var. Partial Differential Equations. 13:1 (2001), 97–121.

[P2] M.R. Pakzad, On topological singular set of maps with finite 3-energy into S^3, Z. Anal. Anwendungen 21:3 (2002), 561–569.

[P3] M.R. Pakzad, Weak density of smooth maps in $W^{1,1}(M,N)$ for non-abelian $\pi_1(N)$, preprint (2000).

[R] T. Rivière, Dense subsets of $H^{\frac{1}{2}}(S^2,S^1)$, Annal. Glob. Anal. and Geom. 18 (2000), 517–528.

[SU] R. Schoen, K. Uhlenbeck, Boundary regularity and the Dirichlet problems in harmonic maps, J. Diff. Geom. 18 (1983), 253–268.

[W] B. White, Rectifiability of flat chains, Ann. Math. (2) 150:1 (1999), 165–184.

[Wh] G.W. Whitehead, Elements of Homotopy Theory, Springer-Verlag, New York, 1978.

Mohammad Reza Pakzad, Max-Planck-Institute for Mathematics in the Sciences, Inselstr. 22 - 26, 04103 Leipzig, Germany pakzad@mis.mpg.de

Tristan Rivière, D-Math, ETH Zentrum, 8092 Zürich, Switzerland riviere@math.ethz.ch

Submitted: September 2001
Revision: November 2002