On topological singular set of maps with finite 3-energy into S^3

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Abstract

We prove that the topological singular set of a map in $W^{1,3}(M,S^3)$ is the boundary of an integer multiplicity rectifiable current in M, where M is a closed smooth manifold of dimension greater than 3. Also we prove that the mass of the minimal integer multiplicity rectifiable current taking this set as the boundary is a strongly continuous functional on $W^{1,3}(M,S^3)$.

1 Introduction

Let M be an oriented smooth closed riemannien manifold of dimension n, and N any closed riemannien manifold isometricly embedded in $\mathbb{R}^N$. Let

$$W^{1,p}(M,N) := \{u \in W^{1,p}(M,\mathbb{R}^N)\,;\, u(x) \in N \quad \text{a.e. on}\, M\}.$$

For $u \in W^{1,p}(M,N)$ the p-energy is given by $E(u) = \int_M |\nabla u|^p dvol_M$.

In [6], F.Bethuel and X.Zheng proved that smooth maps are not strongly dense in $W^{1,p}(M,N)$ if $p < n$ and $\pi_{[p]}(N) \neq 0$, [p] being the integer part of p. In this case, one may want to characterize the maps in $W^{1,p}(M,N)$ which are approximable by smooth maps and identify the obstruction for maps which are not. Precisely, we would like to associate to any map $u \in W^{1,p}(M,N)$ a topological singular set, $\mathbf{S}_u$, which characterizes the approximability of u by smooth maps, i.e. u would be the strong limit of smooth maps if and only if $\mathbf{S}_u = 0$.

In this line, F.Bethuel proved in [3] that $u \in W^{1,2}(\mathbf{B}^n,S^2)$ is strongly approximable by maps in $C^\infty(\mathbf{B}^n,S^2)$, if and only if $d(u^*\omega_{S^2}) = 0$ in the sense of distributions. Here

$\mathbf{B}^n$ is the n-dimensional unit disk. The same result holds for the space $W^{1,p}(\mathbf{B}^n, S^p)$ for any other integer p (See [5]). Thus, the "local" topological obstruction for maps in $W^{1,p}(M, S^{[p]})$ can be defined as a current :

Definition *Let $p < n$ and $u \in W^{1,p}(M, S^{[p]})$. The topological singular set of u, $\mathbf{S}_u \in \mathcal{D}_{n-[p]-1}(M)$, is the current defined by*

$$\mathbf{S}_u(\alpha) := \int_M u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-[p]-1}(M).$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (See[14], 2.2.3) and ω is any $[p]$-form on $S^{[p]}$ for which $\int_{S^{[p]}} \omega = 1$.

Remark 1.1 *Recent developments by F.Hang and F.H.Lin [15] showed that the condition "$\mathbf{S}_u = 0$", though being necessary for the strong approximability of a map $u \in W^{1,p}(M, S^p)$ by smooth maps in this space, is not always sufficient due to some obstructions lying in the "global" topological structure of certain domains. Precisely, there is a map $u \in H^1(\mathbb{CP}^2, S^2)$ for which $d(u^*\omega) = 0$ while u is not in the strong closure of smooth maps in $H^1(\mathbb{CP}^2, S^2)$.*

Two important problems about $\mathbf{S}_u$, $u \in W^{1,p}(M, S^p)$, are still open for almost every integer p. First, we do not know whether $\mathbf{S}_u$ is always the boundary of an integer multiplicity rectifiable current, i.e. if it is an integral flat chain. This has been proved for $p = 1$ or $n - 1$ (See [14], vol II, section 5.4.3) or $p = 2$ (See [19]). The second problem arises if the answer to the first one is positive. Set for $\mathbf{S}$, any integral flat chain in M of dimension k,

$$m_i(\mathbf{S}) := \inf \{\mathbf{M}(\mathbf{T})\,;\, T \in \mathcal{R}_{k+1}(M),\ \partial \mathbf{T} = \mathbf{S}_u\},$$

the minimal mass of integer multiplicity rectifiable currents taking $\mathbf{S}$ as the boundary. Then the question would be to determine whether $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if u_m converges strongly to u in $W^{1,p}(M, S^p)$. The answer is yes for $p = 1$ or $n - 1$, (See [4] and [14], vol II, section 5.4.2), while we do not know whether this is the case for the maps in $H^1(\mathbf{B}^4, S^2)$. We encounter this case when considering the problem of relaxing the Dirichlet energy for maps into S^2. As we saw in [19], generalizing to higher dimensions the algebraic formula given in [4] for the relaxed Dirichlet energy from a 3 dimensional domain into S^2 is possible if we prove that $m_i(\mathbf{S}_u)$ is strongly continuous on $H^1(\mathbf{B}^n, S^2)$.

Another case where the second problem shows its importance is when we try to define a topological singular set for maps in $W^{1,p}(\mathbf{B}^n, N)$. In [5], F.Bethuel, J.M.Coron, F.Demengel and F.Helein gave a description of this set for when N is $([p]-1)$-connected and $\pi_{[p]}(N)$ is torsion free. Considering the problem for when $\pi_{[p]}(N)$ has torsion, the author and T.Rivière remarked that we can define this set as a flat $\pi_{[p]}(N)$-chain if these two questions come to have a positive answer for $[p]$. As an example, the topological singular set of any map in $u \in W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ is a flat $\mathbb{Z}_2$-chain, and is equal to zero if and only if u is a strong limit of smooth maps in $W^{1,1}(\mathbf{B}^n, \mathbb{RP}^2)$ (See [20]).

In this paper we solve these problems for $p = 3$ and 7. The particularity of these two cases reside in the fact that S^3 and S^7 (alongside with S^1) are the only spheres which have this property : There is a smooth multiplication

$$\kappa : S^k \times S^k \to S^k$$

such that the induced homotopic homeomorphism

$$\kappa_* : \pi_k(S^k) \oplus \pi_k(S^k) \to \pi_k(S^k)$$

is the sum of elements in $\pi_k(S^k)$. As a result, the method we use does not work for other values of p. Here is our main result

Theorem 1 *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}(M, S^p)$. Then $\mathbf{S}_u$ is the boundary of an integer multiplicity rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if u_m converges strongly to u in $W^{1,p}(M, S^p)$.*

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If M is not closed we set

$$W^{1,p}_\varphi(M, N) := \{u \in W^{1,p}(M, N)\, ;\, u = \varphi \quad \text{on } \partial M\}$$

where φ is a given boundary data. We assume that φ is in $C^\infty(\partial M, N)$ and can be extended into M by a smooth map. Then we have

Theorem 1 bis *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}_\varphi(M, S^p)$. Then $\mathbf{S}_u$ is the boundary of an integer multiplicity rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if u_m converges strongly to u in $W^{1,p}_\varphi(M, S^p)$.*

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Considering the question of topological singular sets, using the methods of [20], we have these corollaries. The readers may refer to [13], [21] and [20] respectively for definitions and more details.

Corollary 1.1 *Let $\mathbf{B}^n$ be the n-dimensional unit disk, $n > [p] = 3$ or 7, and assume that N is a closed $([p]-1)$-connected riemannien manifold of dimension equal or greater than $[p]$. Then $\mathbf{S}_u$, the topological singular set of any $u \in W^{1,p}(\mathbf{B}^n, N)$, is well defined as a flat $\pi_{[p]}(N)$-chain and the flat norm of $\mathbf{S}_{u_m} - \mathbf{S}_u$ converges to 0 if $u_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$. Moreover u is a strong limit of smooth maps in $W^{1,p}(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$.*

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Remark 1.2 *The cases where N is not $([p]-1)$-connected are more involved. The readers can refer to [16], where T. Rivière and R. Hardt have treated the relatively difficult case of $W^{1,3}(\mathbf{B}^4, S^2)$.*

Corollary 1.1 bis *Let $\mathbf{B}^n$ be the n-dimensional unit disk, $n > [p] = 3$ or 7, and assume that N is a closed $([p]-1)$-connected riemannien manifold of dimension equal or greater than $[p]$. We assume also that $\varphi \in C^\infty(\partial \mathbf{B}^n, N)$ is smoothly extendable into $\mathbf{B}^n$. Then u is a strong limit of smooth maps in $W^{1,p}_\varphi(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$.*

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2 Some known facts

Definition 2.1 *We say that $u \in W^{1,p}(M, S^p)$ is in $R^{\infty,p}(M, S^p)$ if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of M, where $\mathcal{H}^{n-p-1}(B_0) = 0$ and the σ_i , $i = 1, \cdots, m$ are smooth embeddings of the unit disk of dimension $n-p-1$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.*

Theorem 2 (Bethuel,[2]) *$R^{\infty,p}(M, S^p)$ is dense in $W^{1,p}(M, S^p)$ for the strong topology.*

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We recall the definition of $\mathbf{S}_u$, the topological singular set of u :

Definition 2.2 *Let $u \in W^{1,p}(M, S^p)$. We define the current $\mathbf{S}_u \in \mathcal{D}_{n-p-1}(M)$ to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_M u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-p-1}(M). \tag{2.1}$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (See[14], 2.2.3) and ω is some p-form on S^p for which $\int_{S^p} \omega = 1$.

Let ω_1 and ω_2 be two such forms on S^p. We have $\omega_1 - \omega_2 = d\beta$ where β is some smooth 1-form on S^p extendable to $\mathbb{R}^{p+1}$. Let $u \in W^{1,p}(M, S^p)$ and consider a sequence $u_m \in C^\infty(M, \mathbb{R}^{p+1})$ converging to u in $W^{1,p}$. We have

$$u_m^*(d\beta) = d\,(u_m^*\beta)$$

and by passing to the limit, we observe that this holds true for u in the sense of distributions. This proves the independence of $\mathbf{S}_u$ from the choice of ω as we have :

$$d(u^*\omega_1) - d(u^*\omega_2) = du^*(d\beta) = 0$$

in the sense of distributions. Now the existence of the integral (2.1) is a direct consequence of the following inequality :

$$|u^*\omega| \leq \frac{1}{p^{p/2}\alpha_p}|\nabla u|^p \qquad \text{a.e. on } M \tag{2.2}$$

where $\alpha_p := |S^p|$ and $\alpha_p\omega = \omega_V$, is the standard volume form of S^p.

We shall give a description of $\mathbf{S}_u$ for $u \in R^{\infty,p}(M, S^p)$. Clearly if u is smooth a standard operation on pull-back yields

$$d(u^*\omega) = u^*(d\omega) = 0$$

and as a consequence we deduce for $u \in R^{\infty,p}(M, S^p)$ that

$$spt\mathbf{S}_u \subseteq B.$$

Definition 2.3 *Let $u \in R^{\infty,p}(M, S^p)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-p-1)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ let N_a be any $(p+1)$-dimensional smooth submanifold of M, orthogonal to σ_i at a. Consider the embedded $(p+1)$-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the $(p+1)$-vectorfield $\vec{M}_a$ such that $(-1)^{n-p}\vec{\sigma}_i(a) \wedge \vec{M}_a$ is the fixed orientation of M. Then the topological degree of u on the p-dimensional topological sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a and N_a for δ small enough. We call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

Remember that any k-dimensional rectifiable subset $\mathcal{M}$ of M considered with a multiplicity θ and oriented by a unit k-vector field ξ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \xi)(\alpha) := \int_{\mathcal{M}} < \xi, \alpha > \theta\, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(M).$$

We should recall some useful results.

Lemma 2.1 *If u_m is a sequence of maps in $W^{1,p}(M, S^p)$ converging to u, $\mathbf{S}_{u_m}$ tends to $\mathbf{S}_u$ in the sense of currents. That is, for any α, smooth $(n-p-1)$-form in M, we have*

$$\mathbf{S}_u = \lim_{m\to\infty} \mathbf{S}_{u_m}(\alpha).$$

Equivalently

$$m_r(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \quad if \quad u_m \to u \quad in \quad W^{1,p}(M, S^p),$$

where $m_r(\mathbf{S})$ is the minimal mass of normal currents taking $\mathbf{S}$ as their boundary.

■

Lemma 2.2 *Let M be a compact riemannien manifold. Then for any $u \in R^{\infty,p}(M, S^p)$, $\mathbf{S}_u$ is the integer multiplicity rectifiable current $\sum_{i=1}^{m}(deg_{\sigma_i} u)\, \tau(\sigma_i, 1, \vec{\sigma}_i)$. Meanwhile, if ∂M is empty, or if $u|_{\partial M}$ is homotopic to a constant, then $\mathbf{S}_u$ is the boundary of some integer multiplicity rectifiable current of finite mass.*

■

The reader can find the proofs of these statements for the case $p = 2$ in [18] and [19], M being a domain in $\mathbb{R}^n$. The proofs are essentially the same for other values of p and any smooth compact manifold.

Remark 2.1 *By lemma 2.1, theorem 1 would come true for any p if $\frac{m_i(\mathbf{S})}{m_r(\mathbf{S})} < C$, for any integral flat $(n-p-1)$-chain $\mathbf{S}$ in M. The existence of such a constant is an open problem except for when dim $\mathbf{S} = 0, n-2$, where we have the equality $m_i(\mathbf{S}) = m_r(\mathbf{S})$ for any integral flat chain. Refer to [1], [8], [10], [12] and [14], vol II, section 1.3.4 for proofs and different aspects of the problem.*

Theorem 3 (Almgren, Browder and Lieb, [1]) *Let M be as above, $u \in R^{\infty,p}(M, S^p)$, such that either ∂M is empty or $u|_{\partial M}$ is constant, then*

$$m_i(\mathbf{S}_u) \leq \frac{1}{p^{p/2}\alpha_p} \int_M |\nabla u|^p dvol_M$$

■

3 Proof of theorem 1

We identify S^3 (respectively S^7) with the unit spheres in quaternions (respectively Cayley numbers) and observe that they inherit the product structure on these spaces. If we show the quaternion product (respectively Cayley product) by $\kappa(x, y) := x \bullet y$, κ will be a smooth map from $S^k \times S^k \to S^k$, k=3,7, and will satisfy this condition : The induced homotopic homeomorphism

$$\kappa_* : \pi_k(S^k) \oplus \pi_k(S^k) \to \pi_k(S^k)$$

is the sum of elements in $\pi_k(S^k)$. The spheres of dimensions $0, 1, 3$ and 7 are the only spheres for which such κ exist (See [7], section VI.15, p. 412). By $x^{-1} \in S^k$ we mean the right inverse of $x \in S^k$. Set for $u, v \in W^{1,p}(M, S^p)$ and $x \in M$

$$u \bullet v^{-1}(x) := u(x) \bullet v(x)^{-1}.$$

Lemma 3.1 *Let $u, v \in W^{1,p}(M, S^p)$, p=3,7, then $u \bullet v^{-1} \in W^{1,p}(M, S^p)$. Moreover if $\{u_m\}$ is a strongly convergent sequence in $W^{1,p}(M, S^p)$, then $E(u_m \bullet u_k^{-1}) \to 0$ if $m, k \to +\infty$.*

■

Proof : Straight computations show that

$$\nabla(u \bullet v^{-1}) = \nabla u \bullet v^{-1} - u \bullet (v^{-1} \bullet (\nabla v \bullet v^{-1}))$$

which yields

$$|\nabla(u \bullet v^{-1})| \leq |\nabla u| + |\nabla v|$$

as $|u| = |v| = 1$. Thus $u \bullet v^{-1} \in W^{1,p}(M, S^p)$. The smoothness of operations and the Lebesgue dominant convergence yields the second part of lemma. ∎

Lemma 3.2 *If $u, v \in R^{\infty,p}(M, S^p)$, p=3,7, then $u \bullet v^{-1} \in R^{\infty,p}(M, S^p)$ and we have*

$$\mathbf{S}_{u \bullet v^{-1}} = \mathbf{S}_u - \mathbf{S}_v \tag{3.1}$$

.

∎

Proof : That $u \bullet v^{-1} \in R^{\infty,p}(M, S^p)$ is a direct result of smoothness of the product. The relation (3.1) can be deduced from lemma 2.2 and the fact that for any $(n-p-1)$-dimensional face of $B(u \bullet v^{-1})$ we have :

$$deg_\sigma(u \bullet v^{-1}) = deg_\sigma u - deg_\sigma v.$$

∎

Now we present the proof of theorem 1. Let $u \in W^{1,p}(M, S^p)$, p=3,7. By theorem 2 there exists a sequence of maps $u_m \in R^{\infty,p}(M, S^p)$ such that $u_m \to u$ in $W^{1,p}(M, S^p)$. By lemma 3.1, there exist a subsequence u_{m_k} of u_m such that

$$E(u_{m_k} \bullet u_{m_{k+1}}^{-1}) \leq \frac{p^{p/2}\alpha_p}{2^{k+1}}.$$

Meanwhile, using theorem 3 and (3.1), we observe that there is an integer multiplicity rectifiable current $\mathbf{L}_k$ such that

$$\begin{cases} \partial \mathbf{L}_k = \mathbf{S}_{u_{m_k} \bullet u_{m_{k+1}}^{-1}} = \mathbf{S}_{u_{m_k}} - \mathbf{S}_{u_{m_{k+1}}} \\ \\ \mathbf{M}(\mathbf{L}_k) \leq \dfrac{1}{2^k} \end{cases}$$

Choose a finite mass integer multiplicity rectifiable current $\mathbf{L}_0$ such that $\partial \mathbf{L}_0 = \mathbf{S}_{u_{m_1}}$ and put

$$\mathbf{L} := \mathbf{L}_0 - \sum_{i=1}^{+\infty} \mathbf{L}_i.$$

So $\mathbf{M}(\mathbf{L}) < +\infty$ and $\mathbf{L}$ is also an integer multiplicity rectifiable current. Observe that if

$$\mathbf{I}_k := \mathbf{L}_0 - \sum_{i=1}^{k} \mathbf{L}_i,$$

then

$$\partial \mathbf{I}_k = \mathbf{S}_{u_{m_{k+1}}}.$$

Meanwhile $\mathbf{M}(\mathbf{I}_k - \mathbf{L}) \to 0$. This, using lemma 2.1, yields

$$\partial \mathbf{L} = \mathbf{S}_u.$$

(So far we have proved that $\mathbf{S}_u$ is the boundary of some integer multiplicity rectifiable current in M). Moreover,

$$m_i(\mathbf{S}_{u_{m_{k+1}}} - \mathbf{S}_u) \leq \mathbf{M}(\mathbf{I}_k - \mathbf{L}) \to 0 \quad \text{as} \quad k \to +\infty.$$

Consequently, for any convergent sequence $u_m \in R^{\infty,p}(M, S^p)$,

$$m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \tag{3.2}$$

As a result, for any $u \in W^{1,p}(M, S^p)$, $m_i(\mathbf{S}_u) \leq CE(u)$ for $C > 0$ independent of u. Meanwhile, by the strong density of $R^{\infty,p}(M, S^p)$ in $W^{1,p}(M, S^p)$ and lemma 2.1, lemma 3.2 is true for maps in $W^{1,p}(M, S^p)$ too. Using the same method and the proved facts about $\mathbf{S}_u$, we can prove (3.2) for any convergent sequence $u_m \in W^{1,p}(M, S^p)$, i.e.

$$m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0 \quad \text{if} \quad u_m \to u \quad \text{in} \quad W^{1,p}(M, S^p).$$

■

Theorem 1 *bis* is proved following the same ideas.

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References

[1] F.J.Almgren, W.Browder and E.H.Lieb "Co-area, liquid crystals, and minimal surfaces" In Partial differential equations, Lecture Notes in Math., 1306, 1-22. Springer, Berlin, 1988.

[2] F.Bethuel "The approximation problem for Sobolev maps between two manifolds" Acta Math., 167, 153-206, (1992).

[3] F.Bethuel "A characterization of maps in $H^1(B^3, S^2)$ which can be approximated by smooth maps" Ann. Inst. Henri Poincaré, 7, 269-286, (1990).

[4] F.Bethuel, H.Brezis and J.-M.Coron "Relaxed energies for harmonic maps" In Variational Methods, edited by H.Berestycki, J.-M.Coron and J.Ekeland, Birkhäuser, Basel, 1990.

[5] F.Bethuel, J.-M.Coron, F.Demengel and F.Hélein "A cohomological criterion for density of smooth maps in Sobolev spaces between two manifolds" In Nematics, Mathematical and Physical Aspects, edited by J.-M.Coron, J.-M. Ghidaglia and F.Hélein, NATO ASI Series C, 332, 15-23. Kluwer Academic Publishers, Dordrecht, 1991.

[6] F.Bethuel and X.Zheng "Density of smooth functions between two manifolds in Sobolev spaces" J. Funct. Anal. 80, 60-75, (1988).

[7] G.E.Bredon "Topology and Geometry" Springer-Verlag, New York, 1993.

[8] H.Brezis, J.-M.Coron and E.Lieb "Harmonic maps with defects" Comm. Math. Phys., 107, 649 -705, (1986).

[9] F.Demengel and R.Hadiji " Relaxed energy for functionals on $W^{1,1}(B^2, S^1)$ " Nonlin. Anal. Theo., Meth. and Appl., Vol 19, No 7, 625-641, (1992).

[10] L.C.Evans "Partial differential equations and Monge-Kantorovich mass transfer" Current developments in mathematics, 1997.

[11] H.Federer "Geometric Measure Theory" Springer-Verlag, Berlin, 1969.

[12] H.Federer "Real flat chains, cochains and variational problems" Ind. Univ. Math. J. , 24, 351-407, (1974).

[13] W.Fleming "Flat chains over a coefficient group" Trans. Amer. Math. Soc., 121, 160-186, (1966).

[14] M.Giaquinta, G.Modica and J.Soucek " Cartesian Currents in the Calculus of Variations" Springer-Verlag , Berlin, 1998.

[15] F.Hang and F.H.Lin "Topology of Sobolev mappings" Preprint, (2000).

[16] R.Hardt and T.Rivière "Connecting Topological Hopf Singularities", Preprint, (2001).

[17] F.Morgan "Geometric Measure Theory. A Beginner's Guide" Academic Press, Inc., Boston, MA, second edition, 1995.

[18] M.R.Pakzad " Existence of infinitely many weakly harmonic maps from a domain in $\mathbb{R}^n$ into S^2 for non-constant boundary data" Calc. Var., 237, 97-121, (2001).

[19] M.R.Pakzad "Relaxing the Dirichlet energy for maps into S^2 in high dimensions" to appear in Comm. Cont. Math., (2000).

[20] M.R.Pakzad and T.Rivière " Weak density of smooth maps for the Dirichlet energy between manifolds" Preprint, (2000).

[21] B.White "Rectifiability of flat chains" Ann. Math., (2) 150, No.1, 165-184, (1999).