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Existence of infinitely many weakly harmonic maps from a domain in $\mathbb{R}^n$ into S^2 for non-constant boundary data

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Abstract. We prove existence of infinitely many weakly harmonic maps from a domain of $\mathbb{R}^n$ into S^2 for non-constant smooth boundary data.

1 Introduction

Consider the Sobolev space:

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\, \Omega\}$$

where $\Omega \subset \mathbb{R}^n$ is a bounded open set and S^2 is the 2-dimensional unit sphere in $\mathbb{R}^3$. For $u \in H^1(\Omega, S^2)$ the energy $E(u) = \int_\Omega |\nabla u|^2$ is well defined. We call u a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt} E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C_c^\infty(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions:

$$\begin{cases} -\Delta u = u|\nabla u|^2 & \text{in} \quad \Omega \\ u(x) \in S^2 & \text{a.e.} \end{cases}$$

Let $\varphi : \partial\Omega \to S^2$ be a smooth map which has a regular extension into Ω. The existence of a weakly harmonic map equal to φ on the boundary can be easily proved by a straightforward minimizing argument. By the way, the uniqueness and regularity questions for weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ have not the same answers as in the classic cases, i.e. when the target manifold is an euclidean space.

In this paper we consider the question of uniqueness of such extensions. In [10], R. Hardt, D. Kinderlehrer and F.H. Lin had proved the existence of infinitely many weakly harmonic extensions to an axially symmetric boundary condition in $H^1(B^3, S^2)$ where B^3 is the unit ball in $\mathbb{R}^3$. The method consists in constructing a non-axially symmetric harmonic extension and then one obtains infinitely many different harmonic maps with the same boundary data by rotating this extension around the symmetry axis.

Another method consists in finding new harmonic maps by defining new functionals whose critical points are still weakly harmonic. This has been done by F. Bethuel, H. Brezis and J.-M. Coron in [4] where they introduced such functionals which they called “relaxed energies”. Using these functionals they proved for $n = 3$ that if φ is not homotopic to a constant or if

$$\min_{H^1_\varphi(\Omega,S^2)} E(u) < \inf_{C^\infty_\varphi(\Omega,S^2)} E(u)$$

then φ admits infinitely many weakly harmonic extensions inside Ω. Using the same gap condition, T. Isobe proved the corresponding result for the case $n \geq 4$ in [12], still using the relaxed energies whose definition was extended to higher dimensions.

At last, using his *strict dipole insertion lemma*, proved in [14], T. Rivière showed that if Ω is a regular bounded domain of $\mathbb{R}^3$, a non constant smooth boundary data $\varphi : \partial\Omega \to S^2$ admits always infinitely many weakly harmonic extensions (Appeared in [15]). The method, first proposed by F. Bethuel, H. Brezis and J.-M. Coron, consists in producing infinitely many distinct weakly harmonic maps in an inductive process by minimizing the relaxed energies.

The main difficulty in adapting the approach in [15] to higher dimensions is first generalizing the concept of relaxed energies as appeared in [4] to what we will call the F-energies in a suitable way and proving the desired properties for these new energies. Another difficult step consists in finding some equivalent construction in any dimensions of the insertion of 2 singular points with the strict inequality like in [14] for $n = 3$. It appears that ([14], Lemma A.1) can be generalized (via some technical difficulties) by inserting this time $(n-3)$-dimensional singular spheres. Our main result is the following

Theorem 1 *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n \geq 3$, and φ a non-constant smooth map from $\partial\Omega$ into S^2. Then φ admits infinitely many weakly harmonic extensions.* □

Remark 1 This result is independent of the choice of the metric on S^2 . For the details compare with [15].

Remark 2 It seems that the main difficulty to overcome in order to extend the result for p-harmonic maps into S^p, using the same method, is to prove the lower semi-continuity of the generalized relaxed energies which can be defined also in these cases in a natural way.

The paper is organized as follows. In Sect. 2 we recall some elementary facts needed for our work using concepts of Geometric Measure Theory. In Sect. 3 we introduce the F-energies and discuss their characteristics. The readers can refer to [9] for more elaborated discussion of these subjects. Then in Sect. 4 we prove our main result using the strict insertion lemma which we shall prove in the last part of the paper.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^n$, $n \geq 3$, be a bounded open set and let

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\, \Omega\}$$

and

$$H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega, S^2)\,;\, u = \varphi \quad \text{on}\, \partial\Omega\}$$

where φ is a given boundary data. For $u \in H^1_\varphi(\Omega, S^2)$ the Dirichlet energy is given by $E(u) = \int_\Omega |\nabla u|^2$. We assume that φ is in $C^\infty(\partial\Omega, S^2)$ and can be extended into Ω by a smooth map.

2.1 The subspace $R^\infty_\varphi(\Omega, S^2)$

Definition 1 *We say that $u \in H^1_\varphi(\Omega, S^2)$ is in $R^\infty_\varphi(\Omega, S^2)$ if u is smooth except on $B = \bigcup_{i=1}^m \sigma_i \cup B_0$, a compact subset of Ω, where $\mathcal{H}^{n-3}(B_0) = 0$ and the σ_i , $i = 1, \cdots, m$ are smooth embeddings of the unit disk of dimension $n-3$. Moreover we assume that any two different faces of B, σ_i and σ_j, may meet only on their boundaries.*

Remark 3 In ([2], Theorem 2 bis), F. Bethuel has proved that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ for the strong topology.

Definition 2 *Let $u \in H^1_\varphi(\Omega, S^2)$. We define the current $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_\Omega u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega). \tag{1}$$

Here $\mathcal{D}^k(\Omega)$ is the set of smooth k-forms on Ω with compact support (See [9], 2.2.3) and ω is some 2-form on S^2 for which $\int_{S^2}\omega = 1$.

Let ω_1 and ω_2 be two such forms on S^2. We have $\omega_1 - \omega_2 = d\beta$ where β is some smooth 1-form on S^2 extendable to $\mathbb{R}^3$. Let $u \in H^1_\varphi(\Omega, S^2)$ and consider a sequence $u_m \in C^\infty(\Omega, \mathbb{R}^3)$ converging to u in H^1. We have

$$u_m^*(d\beta) = d\left(u_m^*\beta\right)$$

and by passing to the limit, we observe that this holds true for u in the sense of distributions. This proves the independence of $\mathbf{S}_u$ from the choice of ω as we have:

$$d(u^*\omega_1) - d(u^*\omega_2) = du^*(d\beta) = 0$$

in the sense of distributions. Now the existence of the integral (1) is a direct consequence of the following inequality:

$$|u^*\omega| \leq \frac{1}{8\pi}|\nabla u|^2 \qquad \text{a.e. on } \Omega \tag{2}$$

where $4\pi\omega = \omega_V$ is the standard volume form of S^2.

We shall give a description of $\mathbf{S}_u$ for $u \in R^\infty_\varphi(\Omega, S^2)$. Clearly if u is smooth a standard operation on pull-back yields

$$d(u^*\omega) = u^*(d\omega) = 0$$

and as a consequence we deduce for $u \in R^\infty_\varphi(\Omega, S^2)$ that

$$spt\mathbf{S}_u \subseteq B.$$

Definition 3 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and let $B = \bigcup \sigma_i \cup B_0$ be the singular set of u. Suppose that each σ_i is oriented by a smooth $(n-3)$-vectorfield $\vec{\sigma}_i$. For $a \in \sigma_i$ let N_a be the 3-dimensional plane orthogonal to σ_i at a. Consider the 3-disk $M_{a,\delta} = B_\delta(a) \cap N_a$ oriented by the 3-vector $\vec{M}_a$ such that $\vec{\sigma}_i(a) \wedge \vec{M}_a = (-1)^n \vec{\xi}_{\mathbb{R}^n}$. Then the topological degree of u on the 2 dimensional sphere $\Sigma_{a,\delta} = \partial M_{a,\delta}$ is well defined and is independent of the choice of a for δ small enough. We call this integer the degree of u on σ_i and denote it by*

$$deg_{\sigma_i} u\,.$$

Our first goal is to show that for $u \in R^\infty_\varphi(\Omega, S^2)$, $\mathbf{S}_u$ is the integer multiplicity rectifiable current $\sum_{i=1}^m (deg_{\sigma_i} u)\,\tau(\sigma_i, 1, \vec{\sigma}_i)$. Recall that any k-dimensional rectifiable subset $\mathcal{M}$ of $\mathbb{R}^n$ considered with a multiplicity θ and oriented by a unit k-vector field ξ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \xi)(\alpha) := \int_{\mathcal{M}} < \xi, \alpha > \theta\, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(\mathbb{R}^n).$$

Lemma 1 *Let* $\omega = \frac{1}{4\pi}\omega_V$ *and* $u \in R^{\infty}_{\varphi}(\Omega, S^2)$. *Then the* $(n-2)$*-vectorfield* $\vec{D}(u)$ *defined on* $\Omega\backslash B$ *by the equation*

$$< \vec{D}(u)(x), \Psi > \omega_{\mathbb{R}^n} := u^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda^{n-2}(\mathbb{R}^n) \tag{3}$$

is a simple $(n-2)$*-vectorfield tangent to the smooth manifold* $u^{-1}(y)$ *for all regular value* $y = u(x) \in S^2$. *Meanwhile*

$$|\vec{D}(u)| = \frac{1}{4\pi}|J_2 u| \quad \textit{a.e. on } \Omega. \tag{4}$$

□

Remember that an element of $\Lambda_k(\mathbb{R}^n)$ is called *simple* if and only if it equals the exterior product of k vectors of $\mathbb{R}^n$ ([7], 1.6.1).

Proof. Write

$$u^*\omega = \sum_{i<j} u_{ij}\, dx^i \wedge dx^j \quad \text{a.e. on } \Omega$$

and $u_{ij} = 0$ for $i \geq j$. For almost all $x \in \Omega$, $u^*\omega(x)$ is in $\Lambda^2(\mathbb{R}^n)$. Using (3) a short calculation shows that

$$\vec{D}(u)(x) = \sum_{\sigma\in S_n} \frac{1}{(n-2)!} u_{\sigma(1),\sigma(2)} \frac{\partial}{\partial x^{\sigma(3)}} \wedge \cdots \wedge \frac{\partial}{\partial x^{\sigma(n)}}$$

and we get

$$\vec{D}(u)(x) \wedge \vec{\eta} =< \vec{\eta}, u^*\omega(x) > \vec{\xi}_{\mathbb{R}^n} \quad \forall \vec{\eta} \in \Lambda_2(\mathbb{R}^n). \tag{5}$$

So if $y \in S^2$ is a regular point for u we have $\vec{D}(u)(x) \neq 0$ and if $\vec{v}$ is any vector tangent to $u^{-1}(y)$ at x by (5) we obtain

$$\vec{D}(u)(x) \wedge \vec{v} \wedge \vec{w} =< \vec{v} \wedge \vec{w}, u^*\omega(x) > \vec{\xi}_{\mathbb{R}^n} \quad \forall \vec{w} \in \Lambda_1(\mathbb{R}^n)$$

and since $Du(x) \cdot \vec{v} = 0$

$$< \vec{v} \wedge \vec{w}, u^*\omega(x) >= \frac{1}{4\pi} < \Lambda_2(Du)(x) \cdot (\vec{v} \wedge \vec{w}), \omega(y) >$$

$$= \frac{1}{4\pi} < Du(x) \cdot \vec{v} \wedge Du(x) \cdot \vec{w}, \omega(y) >= 0$$

Therefore $\vec{D}(u)(x) \wedge \vec{v} = 0$ for any $\vec{v}$ tangent to $u^{-1}(y)$ at x and as a result $\vec{D}(u)(x)$ is a simple $(n-2)$-vector associated to tangent space of $u^{-1}(y)$

at x (See [7], 1.6.1). Now using (5) and by duality we get (4) as $\omega = \frac{1}{4\pi}\omega_V$ and so

$$|\vec{D}(u)(x)| = |u^*\omega(x)| = \frac{1}{4\pi}|\Lambda_2(Du)(x)|$$
$$= \frac{1}{4\pi}|J_2 u(x)| \quad \text{a.e. on } \Omega. \qquad \square$$

For any $y \in S^2$, regular value of $u \in R^\infty_\varphi(\Omega, S^2)$, we define the current

$$\mathbf{T}^u_y := \tau\left(u^{-1}(y), 1, \frac{\vec{D}(u)}{|\vec{D}(u)|}\right). \tag{6}$$

Proposition 1 *Consider $u \in R^\infty_\varphi(\Omega, S^2)$ and $\mathbf{T}^u_y$ as in (6), then for almost all $y \in S^2$, $\mathbf{T}^u_y$ is a rectifiable current in $\mathbb{R}^n$ with support in $\overline{\Omega}$ and*

$$\partial\mathbf{T}^u_y = \mathbf{S}_u + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{7}$$

where the $(n-3)$-vectorfield $\vec{D}(\varphi)$ on $\partial\Omega$ is defined by the equation

$$< \vec{D}(\varphi)(x), \Psi > \omega_{E_x} := \varphi^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda_{n-3}(E_x)$$

where $E_x = T_x(\partial\Omega)$ is the tangent space to $\partial\Omega$ at x and ω_{E_x} is its unit volume form. $\square$

Proof. First observe that by Sard's theorem, for almost all $y \in S^2$, $u^{-1}(y)$ is a countable union of smooth submanifolds supported in $\overline{\Omega}$. Moreover by Lemma 1, $\frac{\vec{D}(u)}{|\vec{D}(u)|}$ is associated to the tangent space of $u^{-1}(y)$. So by co-area formula we have

$$\int_{S^2} \mathbf{M}(\mathbf{T}^u_y)\, dy = \int_\Omega |J_2 u| \le \frac{1}{2}\int_\Omega |\nabla u|^2$$

and we deduce that $\mathbf{M}(\mathbf{T}^u_y) < +\infty$ for almost all y, i.e. $\mathbf{T}^u_y$ is rectifiable. The claim about $\partial\mathbf{T}^u_y$ is proved in 4 steps:

(i) We prove that $\partial\mathbf{T}^u_y$ is a flat chain.
(ii) We give an expression for $\partial\mathbf{T}^u_y$ of the form

$$\sum_i r^i_y\, \tau(\sigma_i, 1, \vec{\sigma}_i) + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right)$$

using the constancy theorem.
(iii) We prove that $r^i_y = deg_{\sigma_i} u$.

(iv) At last (7) would be proved using the definition of $\mathbf{S}_u$ and the co-area formula.

Step (i): Since u is smooth on $\Omega\backslash B$ we observe that

$$\operatorname{spt}(\partial \mathbf{T}^u_y) \subseteq \partial\Omega \cup B \tag{8}$$

if y is a regular value for u. We know that u is smooth near $\partial\Omega$ and we have $u^{-1}(y) \cap \partial\Omega = \varphi^{-1}(y)$, $\vec{\xi}_{\mathbb{R}^n} = (-1)^{n-1}\vec{\xi}_{E_x} \wedge \vec{n}$ for all $x \in \partial\Omega$. Using (5) for $\vec{D}(u)$ and $\vec{D}(\varphi)$ we get that

$$\vec{D}(u) = (-1)^{n-1}\vec{D}(\varphi) \wedge \vec{n}_{ext} \quad \text{for regular points} \quad x \in \partial\Omega$$

when $\vec{n}_{ext}$ is the outward unit tangent vector to $u^{-1}(y)$ at x. So considering the rules of orientation of manifolds we get

$$\partial \mathbf{T}^u_y \llcorner \partial\Omega = \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{9}$$

which is a rectifiable current for the regular values of u and φ.

For proving the claim we put $\mathbf{S}_y = \partial \mathbf{T}^u_y - \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right)$ and consider the set

$$B_\varepsilon = \{x|\, d(x,B) < \varepsilon\}$$

the ε-neighborhood of B in Ω. By (8) and (9) we get

$$\partial(\mathbf{T}^u_y \llcorner B_\varepsilon) = \mathbf{T}^u_y \llcorner \partial B_\varepsilon + \mathbf{S}_y \quad , \quad \operatorname{spt}\mathbf{S}_y \subseteq B. \tag{10}$$

Since u is smooth on ∂B_ε , $\mathbf{T}^u_y \llcorner \partial B_\varepsilon$ is an $(n-3)$-dimensional normal current. Now using the co-area formula we get

$$\int_{S^2} \mathbf{M}(\mathbf{T}^u_y \llcorner B_\varepsilon)\, dy = \int_{B_\varepsilon} |J_2 u| \le \frac{1}{2}\int_{B_\varepsilon} |\nabla u|^2 \to 0 \quad \text{as} \quad \varepsilon \to 0.$$

So for almost all $y \in S^2$, $\mathbf{M}(\mathbf{T}^u_y \llcorner B_\varepsilon) \to 0$. By (10) we deduce that $\mathbf{S}_y$ is a flat chain as it is a flat-norm limit of normal currents $\mathbf{T}^u_y \llcorner \partial B_\varepsilon$.

Step (ii): $\mathbf{S}_y$ is a flat chain in Ω without boundary. By the Constancy Theorem ([9], 5.3.1, Theorem 3) applied successively to the σ_i, there exist real numbers r^i_y such that

$$\operatorname{spt}(\mathbf{S}_y - r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)) \subseteq \Omega\backslash\sigma_i \quad i = 1, \cdots, m$$

and as a result

$$\operatorname{spt}\left(\mathbf{S}_y - \sum_{i=1}^{m} r^i_y \tau(\sigma_i, 1, \vec{\sigma}_i)\right) \subseteq \Omega\backslash \bigcup_{i=1}^{m} \sigma_i\,.$$

Meanwhile $B = \bigcup_i \sigma_i \cup B_0$ where $\mathcal{H}^{n-3}(B_0) = 0$. So since the support of $\mathbf{S}_y$ lies in B, $\mathbf{S}_y - \sum_{i=1}^m r_y^i \tau(\sigma_i, 1, \vec{\sigma}_i)$ is an $(n-3)$-dimensional flat chain supported in B_0, therefore

$$\mathbf{S}_y = \sum_{i=1}^m r_y^i \tau(\sigma_i, 1, \vec{\sigma}_i)$$

and so

$$\partial \mathbf{T}_y^u = \sum_i r_y^i \, \tau(\sigma_i, 1, \vec{\sigma}_i) + \tau \left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|} \right). \tag{11}$$

Step (iii): We begin this part by proving the following lemma.

Lemma 2 *Let M be a 3-dimensional smooth manifold supported in Ω oriented by $\vec{M}$ a smooth 3-vectorfield. Let $\mathbf{M} = \tau(M, 1, \vec{M})$ and $\Sigma = \partial \mathbf{M}$. Then for almost all $y \in S^2$,*

$$\mathbf{k}(\partial \mathbf{T}_y^u, \mathbf{M}) = (-1)^n \int_\Sigma u^* \omega$$

where $\mathbf{k}(\mathbf{S}, \mathbf{T})$ is the kronecker index of $\mathbf{S}$ and $\mathbf{T}$ as defined in ([9], vol. 1, 5.3.4). $\square$

Proof. For almost all $y \in S^2$ regular value for $(u|_\Sigma)$ (11) is valid and Σ transversally intersects $u^{-1}(y)$ at each point of their intersection. So we have:

$$\int_\Sigma u^* \omega = \sum_{x \in \Sigma \cap u^{-1}(y)} < \vec{\Sigma}(x), \frac{u^*\omega(x)}{|u^*\omega(x)|} >= \mathbf{k}(\mathbf{T}_y^u, \Sigma) \tag{12}$$

Consider the translation $\tau^a : \mathbb{R}^n \to \mathbb{R}^n,\ \tau^a(x) = x + a$. Considering the definition of the kronecker index and ([9], 5.3.4, Theorem 2) we observe that there exists a small enough such that

(i) spt $\tau^a_\# \Sigma \subset \Omega \backslash B$, spt $\tau^a_\# \mathbf{M} \subset \Omega$,

(ii) $\mathbf{T}_y^u \cap \tau^a_\# \Sigma$, $\mathbf{T}_y^u \cap \tau^a_\# \mathbf{M}$ and $\partial \mathbf{T}_y^u \cap \tau^a_\# \mathbf{M}$ exist,

(iii) $\mathbf{k}(\mathbf{T}_y^u, \Sigma) = \mathbf{k}(\mathbf{T}_y^u, \tau^a_\# \Sigma)$, $\mathbf{k}(\partial \mathbf{T}_y^u, \mathbf{M}) = \mathbf{k}(\partial \mathbf{T}_y^u, \tau^a_\# \mathbf{M})$ and

(iv) $\partial(\mathbf{T}_y^u \cap \tau^a_\# \mathbf{M}) = \partial \mathbf{T}_y^u \cap \tau^a_\# \mathbf{M} + (-1)^{n-3} \mathbf{T}_y^u \cap \tau^a_\# \Sigma$.

Therefore by (12)

$$\begin{aligned} \mathbf{k}(\partial \mathbf{T}_y^u, \mathbf{M}) &= \mathbf{k}(\partial \mathbf{T}_y^u, \tau^a_\# \mathbf{M}) = (\partial \mathbf{T}_y^u \cap \tau^a_\# \mathbf{M})(1) \\ &= (-1)^n (\mathbf{T}_y^u \cap \tau^a_\# \Sigma)(1) = (-1)^n \mathbf{k}(\mathbf{T}_y, \tau^a_\# \Sigma) \\ &= (-1)^n \mathbf{k}(\mathbf{T}_y^u, \Sigma) = (-1)^n \int_\Sigma u^* \omega. \end{aligned}$$

which proves the lemma. □

Now take $\mathbf{M}_{a,\delta} = \tau(M_{a,\delta}, 1, \vec{M}_{a,\delta})$ as in the definition 3. Applying Lemma 2 and (12) to $\mathbf{M}_{a,\delta}$ we get:

$$deg_{\sigma_i} u = \int_{\Sigma_{a,\delta}} u^*\omega = (-1)^n \mathbf{k}(\partial \mathbf{T}^u_y, \mathbf{M}_{a,\delta}) = r^i_y\,. \tag{13}$$

Step (iv): Let $\alpha \in \mathcal{D}^{n-3}(\Omega)$. By the co-area formula and (4) we get:

$$\begin{aligned}\int_\Omega u^*\omega \wedge d\alpha &= \frac{1}{4\pi}\int_{S^2} dy \int_{u^{-1}(y)} \frac{u^*\omega \wedge d\alpha}{|u^*\omega|} \\ &= \frac{1}{4\pi}\int_{S^2} dy \int_{u^{-1}(y)} < \frac{\vec{D}(u)}{|\vec{D}(u)|}, d\alpha > \, d\mathcal{H}^{n-2} \\ &= \frac{1}{4\pi}\int_{S^2} \mathbf{T}^u_y(d\alpha)\, dy = \frac{1}{4\pi}\int_{S^2} \partial\mathbf{T}^u_y(\alpha)\, dy\end{aligned}$$

and since $\alpha|_{\partial\Omega} = 0$ using (11) and (13) we obtain:

$$\begin{aligned}\int_\Omega u^*\omega \wedge d\alpha &= \frac{1}{4\pi}\int_{S^2} dy \sum_{i=1}^m (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i)(\alpha) \\ &= \sum_{i=1}^m (deg_{\sigma_i} u)\, \tau(\sigma_i, 1, \vec{\sigma}_i)(\alpha)\end{aligned} \tag{14}$$

which completes the proof of Proposition 1 regarding the definition of $\mathbf{S}_u$ and the formula for $\partial\mathbf{T}^u_y$ in (11). □

Corollary 1 *Let $u \in R^\infty_\varphi(\Omega, S^2)$ and $B = \bigcup_i \sigma_i \cup B_o$ its singular set. Then*

$$\mathbf{S}_u = \sum_i (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i).$$

□

Proof. Refer to the relation (14) in the proof of Proposition 1. □

3 The F-energy on $H^1_\varphi(\Omega, S^2)$

In this section we define for any $v \in H^1_\varphi(\Omega, S^2)$ a functional F_v on $H^1_\varphi(\Omega, S^2)$ which has two interesting properties. First, it is lower semi-continuous and second, its critical points are also the critical points of the energy E, i.e. the critical points of F, in particular its minimizers, would be weakly harmonic maps. In fact this "F-energy" is a natural generalization of the "relaxed energy" in dimension 3 introduced in [4], except that in higher dimensions the functional F may not be a relaxed energy for $H^1_\varphi(\Omega, S^2)$: i.e. there exist cases where

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \min_{C^\infty_\varphi(\Omega,S^2)} E\,,$$

(See [13]).

Definition 4 *Let u, v be two maps in $H^1_\varphi(\Omega, S^2)$. We define the connection between u and v to be*

$$L(u,v) = \sup_{\substack{\psi \in \Omega^\infty_{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \le 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_\Omega v^*\omega \wedge d\psi \right\} \tag{15}$$

where ω is any 2-form on S^2 with $\int_{S^2} \omega = 1$. We will often take $\omega = \frac{1}{4\pi}\omega_V$ which is more suitable for computations.

Remark 4 We recall that the mass of currents is in fact the dual of the comass norm of differential forms (See [7], 4.1.7). So, from Geometric Measure Theory point of view, it would be more natural to use the comass norm of $d\psi$ instead of its euclidean norm in the definition of L. Meanwhile the euclidean norm is preferred for the relative simplicity of the proof of lower semi-continuity of F_v .

Proposition 2 *We have the following inequality:*

$$L(u,v) \le C\|\nabla u - \nabla v\|_2(\|\nabla u\|_2 + \|\nabla v\|_2) \quad \forall u, v \in H^1_\varphi(\Omega, S^2). \tag{16}$$

$\square$

Proof. We write

$$d\psi = \sum_{1\le i_3<i_4<\cdots<i_n\le n} \psi_{i_3 i_4 \cdots i_n} dx^{i_3} \wedge dx^{i_4} \wedge \cdots \wedge dx^{i_n}$$

and we have

$$\sum_{i_3<i_4<\cdots<i_n} |\psi_{i_3 i_4\cdots i_n}|^2 = |d\psi|^2 \le 1.$$

Now by simple calculations we obtain:

$$< \vec{\xi}_{\mathbb{R}^n}, u^*\omega \wedge d\psi >= \frac{1}{8\pi} \sum_{\substack{i_3 < i_4 < \cdots < i_n \\ \{i_1, \cdots, i_n\} = \{1, \cdots, n\}}} u \cdot (u_{x^{i_1}} \wedge u_{x^{i_2}})\, \psi_{i_3 i_4 \cdots i_n}$$

and the proposition is proved using the same method used in [4], Theorem 3. □

Now let $u \in H^1_\varphi(\Omega, S^2)$ and for $u_0 \in C^\infty_c(\Omega, S^2)$ consider the variation $u(t) = \frac{u+tu_0}{|u+tu_0|}$. As a consequence for t small enough $u(t) \in H^1_\varphi(\Omega, S^2)$ and we have:

Lemma 3 *For all $u, v \in H^1_\varphi(\Omega, S^2)$ and for t small enough $L(u(t), v) = L(u, v)$.* □

Proof. Pay attention that if $u_n \to u$ in H^1 then for t small enough we have $u_n(t) \to u(t)$ in H^1. So in the view of the Proposition 2 and by using the fact that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ (See Remark 3), it suffices for us to prove this lemma for $u, v \in R^\infty_\varphi(\Omega, S^2)$. For such u and v we get by the co-area formula and Proposition 1:

$$\begin{aligned}\int_\Omega u^*\omega \wedge d\psi - \int_\Omega v^*\omega \wedge d\psi &= \frac{1}{4\pi}\int_{S^2} (\mathbf{T}^u_y - \mathbf{T}^v_y)(d\psi)\, dy \\ &= (\mathbf{S}_u - \mathbf{S}_v)(\psi). \end{aligned} \tag{17}$$

Meanwhile for $u \in R^\infty_\varphi(\Omega, S^2)$, using the corollary 1, we have $\mathbf{S}_u = \mathbf{S}_{u(t)}$ as u and $u(t)$ have the same singular set and the same degrees on its components. By (17) we get:

$$\begin{aligned} L(u(t), v) &= \sup_{|d\psi|_\infty \le 1} \left(\mathbf{S}_{u(t)} - \mathbf{S}_v\right)(\psi) \\ &= \sup_{|d\psi|_\infty \le 1} \left(\mathbf{S}_u - \mathbf{S}_v\right)(\psi) = L(u, v) \end{aligned}$$

and the lemma is proved. □

Proposition 3 *For fixed $v \in H^1_\varphi(\Omega, S^2)$ let*

$$F_v(u) := E(u) + 8\pi L(u, v).$$

Then F_v is a lower semi-continous functional on $H^1_\varphi(\Omega, S^2)$ and its critical points are weakly harmonic maps. □

Remark 5 T. Isobe has proved the lower semi-continuity of the functionals

$$F_{\psi,\lambda}(u) = E(u) + 8\pi\lambda\left\{\int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi\right\}$$

for $\lambda < C(n)$, $n \geq 4$ (See [12]). But what we need here is the same result for $\lambda = 1$ for which we have to prefer another argument.

Proof. Again as in the Proposition 2, the proof of lower semi-continuity of F_v is the same as the proof of lower semi-continuity of the relaxed energy in [4]. Using Lemma 3 we obtain

$$\frac{d}{dt}F_v(u(t))_{|t=0} = \frac{d}{dt}E(u(t))_{|t=0} + 8\pi\frac{d}{dt}L(u(t), v)_{|t=0} = \frac{d}{dt}E(u(t))_{|t=0}$$

so as a result the critical points of F_v are those of E. □

4 The existence of infinitely many weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ for non-constant boundary datas

We shall state here the main result of the paper.

Theorem 1 *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n \geq 3$, and φ a non-constant smooth map from $\partial\Omega$ to S^2. Then φ admits infinitely many weakly harmonic extensions.* □

For proving this theorem we apply a method proposed by F. Bethuel, H. Brezis and J.-M. Coron which uses the F-energy as an efficient tool for finding the new weakly harmonic maps and a technical lemma which we shall prove in the following section.

Lemma 4 *Let Ω be a bounded regular domain in $\mathbb{R}^n$ and u a regular non-constant map from Ω to S^2. Let x_0 be a point of Ω for which $\nabla u(x_0) \neq 0$. Then for every $\rho > 0$ there exists a map $v \in H^1(\Omega, S^2)$ and $0 < \delta < \rho$ such that*

(i) v=u on $\Omega \backslash B_\rho(x_0)$
(ii) $\mathbf{S}_v = \tau(\sigma, 1, \vec{\sigma})$
(iii) $E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2} = E(u) + 8\pi L(v, u)$

where σ is an $(n-3)$-dimensional sphere of center x_0 and radius δ and ω_k is the volume of the unit k-dimensional disk. □

This lemma, called the strict insertion of singularities, was firstly proved for the case $n = 3$ by T. Rivière in [14]. The computations used rely on the previous computations for inserting coverings of S^2 in dimension 2 (See [5]). The axially symmetric version of it was proved in [11].

Proof of Theorem 1. Two situations may take place:

(1) There are infinitely many distinct minimizers for E in $H^1_\varphi(\Omega, S^2)$ and so the problem is solved.
(2) There are only a finite number of minimizers for E on $H^1_\varphi(\Omega, S^2)$.

In this case let $w_1, \cdots, w_m$ be the minimizing maps. By the partial regularity theory of [16] and considering the fact that φ is not constant we deduce the existence of Ω_1, an open subset of Ω, on which w_1 is smooth and some $x_0 \in \Omega_1$ for which $\nabla w_1(x_0) \neq 0$. For some $\rho > 0$ which will be fixed later we apply the Lemma 4 to w_1 on Ω_1 and name the transformed map v_1. So we have

$$E(v_1) < E(w_1) + 8\pi L(v_1, w_1). \tag{18}$$

Now suppose that u_1 is a minimizing map for F_{v_1} on $H^1_\varphi(\Omega, S^2)$. By Proposition 4 such maps exist and are weakly harmonic. We shall prove that for ρ sufficiently small u_1 is different from all the w_i. We distinguish two cases:

(a) $L(w_k, w_1) = 0$: By (18) we obtain

$$F_{v_1}(u_1) \leq F_{v_1}(v_1) = E(v_1) < E(w_1) + 8\pi L(v_1, w_1). \tag{19}$$

Moreover subadditionality of L gives

$$|L(v_1, w_1) - L(v_1, w_k)| \leq L(w_k, w_1) = 0, \tag{20}$$

so $L(v_1, w_1) = L(v_1, w_k)$ and using the fact that $E(w_1) = E(w_k)$, (19) implies

$$F_{v_1}(u_1) < F_{v_1}(w_k).$$

This strict inequality proves naturally that $u_1 \neq w_k$ when $L(w_1, w_k) = 0$.

(b) $L(w_k, w_1) > 0$: We have

$$L(w_k, v_1) + L(v_1, w_1) \geq L(w_k, w_1), \tag{21}$$

meanwhile by the Lemma 4

$$L(v_1, w_1) = \omega_{n-2}\delta^{n-2} < \omega_{n-2}\rho^{n-2}, \tag{22}$$

thus

$$\begin{aligned} F_{v_1}(w_k) &= E(w_k) + 8\pi L(w_k, v_1) \\ &\geq E(w_1) + 8\pi(L(w_k, w_1) - \omega_{n-2}\rho^{n-2}). \end{aligned} \tag{23}$$

Now it is sufficient to choose $\rho > 0$ such that for all w_k verifying $L(w_k, w_1) > 0$ we have the inequality

$$0 < \omega_{n-2}\rho^{n-2} < \frac{L(w_k, w_1)}{2}, \tag{24}$$

then by (22) we have $L(w_k, w_1) - \omega_{n-2}\rho^{n-2} > \omega_{n-2}\rho^{n-2} > L(v_1, w_1)$ and this, added to (23) implies:

$$F_{v_1}(w_k) > F_{v_1}(w_1) \geq F_{v_1}(u_1),$$

which combined with part (a) proves that u_1 is different from all the w_k .

We construct by induction a sequence u_j of distinct weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ which are also different from the w_i, using the same method. Choose ρ_{j+1} such that

$$\begin{cases} 0 < \omega_{n-2}\rho_{j+1}^{n-2} < Min\left\{\dfrac{L(w_k, w_1)}{2};\right. \\ \qquad\qquad\qquad\qquad \left.\text{for } k \text{ verifying} \quad L(w_k, w_1) > 0\right\} \\ and \\ 0 < \omega_{n-2}\rho_{j+1}^{n-2} < Min\left\{\dfrac{E(u_i) - E(w_1)}{8\pi}; \quad i = 1, \cdots, j\right\} \end{cases} \tag{25}$$

Let u_{j+1} be a minimizer of $F_{v_{j+1}}$ when v_{j+1} is the transformed map of w_1 on $B_{\rho_{j+1}}(x_0)$ as in Lemma 4. Again the first inequality in (25) assures that u_{j+1} is distinct from the w_i. For seeing that $u_{j+1} \neq u_i$ for $i \leq j$, using the strict inequality of Lemma 4 we observe that

$$F_{v_{j+1}}(u_{j+1}) \leq F_{v_{j+1}}(v_{j+1}) = E(v_{j+1}) < E(w_1) + 8\pi\omega_{n-2}\rho_{j+1}^{n-2}. \quad (26)$$

Moreover from (25) we have

$$8\pi\omega_{n-2}\rho_{j+1}^{n-2} < E(u_i) - E(w_1). \quad (27)$$

Thus combining (26) and (27) imply that $E(u_{j+1}) \leq F_{v_{j+1}}(u_{j+1}) < E(u_i)$. This yields that $u_{j+1} \neq u_i$ for $i \leq j$ and completes the proof of the theorem. □

5 The strict insertion of a singular sphere

We would follow the method used by T. Rivière in [14] for the case $n = 3$.

5.1 Notations

We replace x_0 by 0 using a suitable translation in $\mathbb{R}^n$. We choose also an orthonormal basis $(\vec{i}, \vec{j}, \vec{k}_1, \cdots, \vec{k}_{n-2})$ for $\mathbb{R}^n$ such that

$$u_x(0) \neq 0, \quad u_x(0) \cdot u_y(0) = 0. \quad (28)$$

(See [5]). Let $(x, y, z_1, \cdots, z_{n-2})$ be the coordinates in the new basis. We introduce also the polar coordinates (r, θ) , $(R, \theta_1, \cdots, \theta_{n-4}, \varphi)$ as follows

$$\begin{cases} x = r\cos\theta \\ y = r\sin\theta \\ z_1 = R\cos\theta_1 \\ z_2 = R\sin\theta_1\cos\theta_2 \\ . \\ . \\ . \\ z_{n-3} = R\sin\theta_1\cdots\sin\theta_{n-4}\cos\varphi \\ z_{n-2} = R\sin\theta_1\cdots\sin\theta_{n-4}\sin\varphi \end{cases} \quad (29)$$

where $0 \leq \theta_i \leq \pi$, $0 \leq \varphi \leq 2\pi$ and $|\mathbf{z}| = R$ for $\mathbf{z} = (z_1, \cdots, z_{n-2})$.

Now for δ sufficiently small and $R \in [0, \delta + \delta^2]$ we define two unit vector fields

$$I(\mathbf{z}) = \frac{u_x(0,0,\mathbf{z})}{|u_x(0,0,\mathbf{z})|}, \quad K(\mathbf{z}) = u(0,0,\mathbf{z}). \quad (30)$$

Since u takes its values in S^2, I and K are orthogonal. Let $a = |u_x(0)|$ and $b = |u_y(0)|$. We define $J(\mathbf{z})$ to be a smooth vectorfield such that (I, J, K) form an orthonormal basis. We verify then

$$\begin{aligned} u_x(0,0,\mathbf{z}) &= (a + O(R))I(\mathbf{z}) \\ u_y(0,0,\mathbf{z}) &= O(R)I(\mathbf{z}) + (b + O(R))J(\mathbf{z}). \end{aligned} \tag{31}$$

5.2 Sketch of the proof

We shall transform u in the region

$$C^\delta = \{(x, y, \mathbf{z}) \in \Omega \,|\quad 0 \le R \le \delta + \delta^2, \quad 0 \le r \le 2\delta^2\}.$$

For δ sufficiently small , the transformed map v would be singular exactly on the $(n-3)$-dimensional sphere $\sigma = \{(0, 0, \mathbf{z});\ R = \delta\}$ and will satisfy

$$deg_\sigma v = 1, \quad E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2}. \tag{32}$$

For this aim we define the map u^δ as follows

(a) $u = u^\delta$ outside C^δ

(b) In the region

$$c^\delta = \{(x, y, \mathbf{z}) \,|\, R < \delta - \delta^2, 0 \le r \le 2\delta^2\}$$

u^δ would be an interpolation between u ouside c^δ and a conformal map on each disk centered at $(0, 0, \mathbf{z})$ and of radius δ^2 in the region

$$c_1^\delta = \{(x, y, \mathbf{z}) \,|\, R < \delta - \delta^2\,, 0 \le r \le \delta^2\}$$

exactly as it is described by T. Rivière in [14], following the method of H. Brezis and J.-M. Coron in [5] .

(c) For the region $\tilde{c}^\delta = C^\delta \backslash c^\delta$, u^δ will be the conjugation of the value of u^δ on $\partial\tilde{c}^\delta$ with the projection $\Pi : \tilde{c}^\delta \to \partial\tilde{c}^\delta$ which is defined as follows : For $p \in \tilde{c}^\delta$, $\Pi(p)$ is the intersection with $\partial\tilde{c}^\delta$ of the line orthogonal to σ which passes through p.

It will be showed that $v = u^\delta$ for δ small enough is a desired map. In the last step we will prove that $L(u, v) = \omega_{n-2}\delta^{n-2}$, the volume of the $(n-2)$-disk of the boundary σ.

5.3 The construction of u^δ in c^δ

For $(x, y, \mathbf{z}) \in c^\delta$ we define

(i) If $r < \delta^2$:

$$u^\delta = \frac{2\lambda}{\lambda^2 + r^2}(xI(\mathbf{z}) + yJ(\mathbf{z}) - \lambda K(\mathbf{z})) + K(\mathbf{z}) \tag{33}$$

where $\lambda = c\delta^4$ and c will be fixed later.

(ii) If $\delta^2 \le r \le 2\delta^2$:

$$\begin{aligned} u^\delta &= (A_1 r + B_1)I(\mathbf{z}) + (A_2 r + B_2)J(\mathbf{z}) \\ &\quad + \sqrt{1 - (A_1 r + B_1)^2 - (A_2 r + B_2)^2}\, K(\mathbf{z}) \end{aligned} \tag{34}$$

where A_i and B_i depend only on $\mathbf{z}, \theta, r$ as follows:

$$\begin{cases} 2\delta^2 A_i + B_i = u_i(2\delta^2\cos\theta, 2\delta^2\sin\theta, \mathbf{z}) \\ \text{for } i = 1, 2\, (u_i \text{ is the } i\text{-th coordinate of } u \text{ in } (I(\mathbf{z}), J(\mathbf{z}), K(\mathbf{z})) \\ \delta^2 A_1 + B_1 = \dfrac{2\lambda\delta^2}{\lambda^2 + \delta^4}\cos\theta \\ \delta^2 A_2 + B_2 = \dfrac{2\lambda\delta^2}{\lambda^2 + \delta^4}\sin\theta. \end{cases} \tag{35}$$

5.3.1 The estimates for $E(u^\delta)$ in $c_2^\delta = c^\delta \backslash c_1^\delta$

Following the same computations as in [5] or [14] we have the following estimates on c_2^δ for fixed $\mathbf{z}$:

$$\begin{cases} \displaystyle\int_{\delta^2 \le r \le 2\delta^2} |\nabla_{xy} u^\delta(x, y, \mathbf{z})|^2\, dxdy \\ = 4\pi\delta^4(a^2 + b^2 - 2c^2 + (a^2 + b^2 + 8c^2 - 4ac - 4bc)\ln 2) + O(\delta^5). \\ \left|\dfrac{\partial u^\delta}{\partial z_i}(x, y, \mathbf{z})\right| = \left|\dfrac{\partial u}{\partial z_i}(0, 0, \mathbf{z})\right| + O(\delta^2) \quad \text{for } i = 1, \cdots, n-2. \\ |\nabla u^\delta| \le C \quad \text{for } C > 0 \text{ independent of } \delta. \end{cases} \tag{36}$$

Note that by $\nabla_{xy} u$ we mean the matrix of first partial derivatives of u in x and in y. As a result we have the following estimate for the energy on c_2^δ:

$$\begin{aligned}\int_{c_2^\delta} |\nabla u^\delta|^2 = 4\pi\omega_{n-2}\delta^{n+2} \\ \times(a^2+b^2-2c^2+(a^2+b^2+8c^2-4ac-4bc)\ln 2) \\ +\pi((2\delta)^2-(\delta^2))\int_{0\le R\le\delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2\, d\mathbf{z} \\ +O(\delta^{n+3}). \qquad (37)\end{aligned}$$

5.3.2 The estimates for $E(u^\delta)$ in c_1^δ

Firstly for a fixed $\mathbf{z}$, u^δ is a conformal diffeomorphism from the disk $B^2((0,0,\mathbf{z}),\delta^2)$ into S^2 and we get:

$$\begin{aligned}\int_{r\le\delta^2} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2\, dxdy &= 2\,\mathrm{Area}(u^\delta(B^2((0,0,\mathbf{z}),\delta^2),\mathbf{z})) \\ &= 8\pi - 8\pi c^2\delta^4 + O(\delta^5).\end{aligned} \qquad (38)$$

and by integration on $\mathbf{z}$ we obtain:

$$\begin{aligned}&\int_{c_1^\delta} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2\, dxdydz_1\cdots dz_{n-2} \\ &= \frac{\omega_{n-2}}{(n-2)}\int_0^{\delta-\delta^2} R^{n-3}dR \int_{r\le\delta^2} |\nabla_{xy} u^\delta(x,y,\mathbf{z})|^2\, dxdy \\ &= 8\pi\omega_{n-2}(\delta-\delta^2)^{n-2} - 8\pi\omega_{n-2}c^2\delta^{n+2} + O(\delta^{n+3}). \qquad (39)\end{aligned}$$

Meanwhile we estimate the $\mathbf{z}$-derivatives of u^δ in c_1^δ. Firstly we have

$$\begin{aligned}\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z}) = \frac{2\lambda}{\lambda^2+r^2}\left(x\frac{dI}{dz_i}+y\frac{dJ}{dz_i}-\lambda\frac{dK}{dz_i}\right)+\frac{dK}{dz_i} \\ \text{for } i=1,\cdots,n-2. \qquad (40)\end{aligned}$$

We estimate $\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z})$ in two regions:

(a) $r\le\delta^3$: Using (40) we observe that for $0\le r\le\delta^2$:

$$|\nabla_{z_i} u^\delta| \le \left|\frac{dK}{dz_i}\right| + \frac{2\lambda}{(\lambda^2+r^2)^{\frac12}} \le C \quad \text{independent of } \delta \qquad (41)$$

and as a result

$$\int_{r\leq\delta^3}|\nabla_{\mathbf{z}}u^\delta|^2dxdy=O(\delta^6),\tag{42}$$

which implies

$$\int_{c_1^\delta}|\nabla_{\mathbf{z}}u^\delta|^2=O(\delta^{n+4}).\tag{43}$$

(b) $\delta^3\leq r\leq\delta^2$: We have

$$\left|\frac{2\lambda}{\lambda^2+r^2}(x\frac{dI}{dz_i}+y\frac{dJ}{dz_i})\right|\leq\frac{2\lambda r}{\lambda^2+r^2}\leq C\frac{\lambda}{r}=O(\delta).\tag{44}$$

So using (40)

$$\frac{\partial u^\delta}{\partial z_i}(x,y,\mathbf{z})=\left(\frac{r^2-\lambda^2}{r^2+\lambda^2}\right)\frac{\partial u}{\partial z_i}(0,0,\mathbf{z})+O(\delta)\quad\text{for}\quad\delta^3\leq r\leq\delta^2,$$

and we get

$$\begin{aligned}\int_{\delta^3\leq r\leq\delta^2}|\nabla_{\mathbf{z}}u^\delta|^2\,dxdy&=\left(2\pi\int_{\delta^3}^{\delta^2}\left(\frac{r^2-\lambda^2}{r^2+\lambda^2}\right)^2r\,dr\right)\\&\quad\times|\nabla_{\mathbf{z}}u(0,0,\mathbf{z})|^2+O(\delta^5)\\&=\pi\delta^4|\nabla_{\mathbf{z}}u(0,0,\mathbf{z})|^2+O(\delta^5).\end{aligned}$$

This last estimate combined with (43) yields

$$\begin{aligned}&\int_{c_1^\delta}|\nabla_{\mathbf{z}}u^\delta|^2\\&\quad=\pi\delta^4\int_{0\leq R\leq\delta-\delta^2}|\nabla_{\mathbf{z}}u(0,0,\mathbf{z})|^2\,dz_1\cdots dz_{n-2}+O(\delta^{n+3})\,.\end{aligned}\tag{45}$$

At last combining (39) and (45) we obtain:

$$\begin{aligned}\int_{c_1^\delta}|\nabla u^\delta|^2&=8\pi\omega_{n-2}(\delta-\delta^2)^{n-2}-8\pi\omega_{n-2}c^2\delta^{n+2}\\&\quad+\pi\delta^4\int_{0\leq R\leq\delta-\delta^2}|\nabla_{\mathbf{z}}u(0,0,\mathbf{z})|^2\,d\mathbf{z}+O(\delta^{n+3})\,.\end{aligned}\tag{46}$$

5.3.3 The evaluation of $E(u^\delta)$ on $\tilde{c}^\delta$

As briefly mentioned above in the the sketch of the proof, u^δ in the region $\tilde{c}^\delta$ is defined as follows: We define the projection $h : \tilde{c}^\delta \to \sigma$ by

$$h(x, y, z_1, z_2, \cdots, z_{n-2}) = (0, 0, \frac{\delta z_1}{R}, \cdots, \frac{\delta z_{n-2}}{R}) \tag{47}$$

Then the projection Π , defined on

$$\tilde{c}^\delta = \{(x, y, \mathbf{z}) \,|\delta - \delta^2 \le R \le \delta + \delta^2\,, \quad 0 \le r \le \delta^2\}$$

sends each point p to the intersection between $\partial\tilde{c}^\delta$ and the line passing through p and $h(p)$. We take

$$u^\delta = \left(u^\delta|_{\partial\tilde{c}^\delta}\right) \circ \Pi.$$

Pay attention that the points p and $\Pi(p)$ lie in the 3-plane orthogonal to σ at $h(p)$.

Using the co-area formula we have

$$\int_{\tilde{c}^\delta} |\nabla u^\delta|^2 = \int_\sigma d\mathcal{H}^{n-3} \int_{h^{-1}(w)} \frac{|\nabla u^\delta|^2}{|J_{n-3}h|}\, d\mathcal{H}^3 \tag{48}$$

Moreover $|J_{n-3}h| = \left(\frac{\delta}{R}\right)^{n-3}$ and

$$\begin{aligned} h^{-1}(w) = \{&(x, y, R, \theta_1, \cdots, \theta_{n-4}, \varphi) \\ &\in \tilde{c}^\delta \,|\, \delta - \delta^2 \le R \le \delta + \delta^2\,, 0 \le r \le 2\delta^2\,, \\ &\theta_i = const. \text{ for } i = 1, \cdots, n-4, \text{ and } \varphi = const.\,\} \end{aligned}$$

is a cylinder of the height $2\delta^2$, of radius $2\delta^2$ and of center $w \in \sigma$. We now estimate the value of $\int_{h^{-1}(w)} R^{n-3} |\nabla u^\delta|^2\, dx\, dy\, dR$.

We write $h^{-1}(w)$ as the union of two separate regions G_w and H_w:

(1) $G_w = \Pi^{-1}(\partial c_1^\delta \cap h^{-1}(w))$ is the little 3-cone of vertex w, lying in the plane orthogonal to σ at w, whose end is the disk D_{δ^2} of center $(0, 0, \delta - \delta^2, \theta_1^w, \cdots, \theta_{n-4}^w, \varphi^w)$ and of radius δ^2. Pay attention that on this disk u^δ is the conformal map defined in (33).

(2) H_w is the complementar of G_w in $h^{-1}(w)$: i.e. $H_w = \Pi^{-1}(\partial\tilde{c}^\delta \backslash \partial c_1^\delta \cap h^{-1}(w))$.

See Fig. 1 and Fig. 2 to visualize these regions for $n = 4$. For estimating $|\nabla u^\delta|$ on G_w we proceed by changing the coordinates. Let R' be the distance of the point $p = (x, y, z_1, \cdots, z_{n-2}) \in G_w$ from w, the vertex of the cone, and let x' and y' be the two first coordinates of $\Pi(p)$ in D_{δ^2} (See Fig. 2). We have

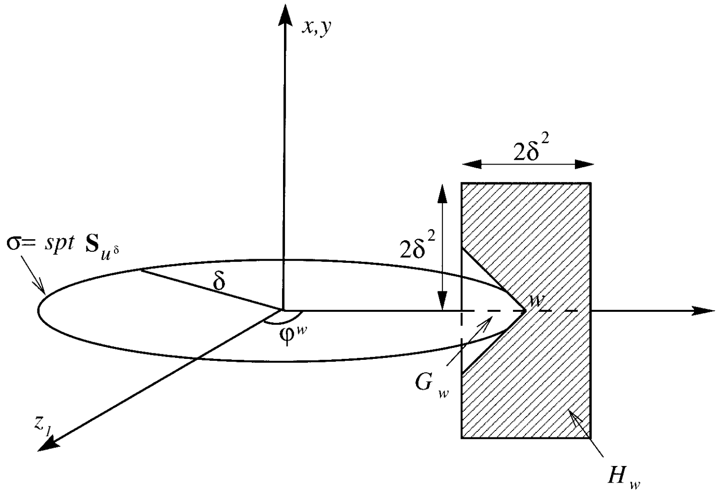
x,y
$2\delta^2$
$2\delta^2$
σ= spt $\mathbf{S}_{u^\delta}$
δ
w
φ^w
G_w
z_1
H_w

Fig. 1

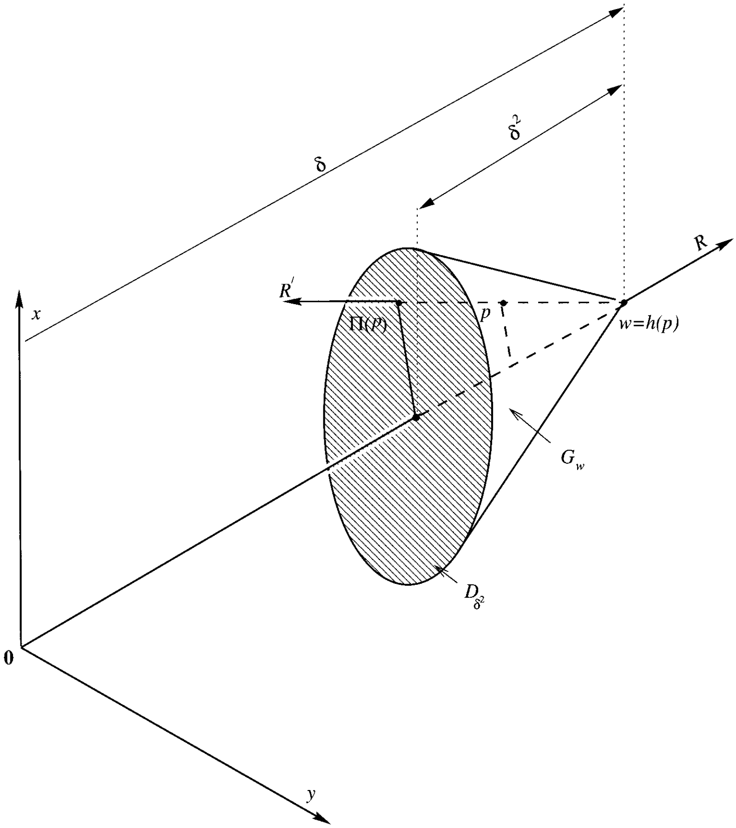
δ
δ^2
R
R'
x
$\Pi(p)$
p
w=h(p)
G_w
D_{δ^2}
0
y

Fig. 2

$$\begin{cases} x' = \dfrac{\delta^2 x}{\delta - R} \\ y' = \dfrac{\delta^2 y}{\delta - R} \\ R' = \sqrt{r^2 + (\delta - R)^2} \\ \theta_i = \theta_i \ , \ \varphi = \varphi \end{cases} \quad \text{and} \quad \begin{cases} x = \dfrac{x' R'}{\sqrt{\delta^4 + r'^2}} \\ y = \dfrac{y' R'}{\sqrt{\delta^4 + r'^2}} \\ \delta - R = \dfrac{\delta^2 R'}{\sqrt{\delta^4 + r'^2}} \end{cases} \tag{49}$$

Now u^δ is constant on the rays passing by w, so we get

$$\begin{aligned} & u^\delta(x', y', R', \theta_1, \cdots, \theta_{n-4}, \varphi) \\ & \quad = u^\delta(x', y', \sqrt{\delta^4 + r'^2}, \theta_1, \cdots, \theta_{n-4}, \varphi) \end{aligned} \tag{50}$$

i.e. $\dfrac{\partial u^\delta}{\partial R'} = 0$. Also by a simple calculation of the derivatives using (49) we have for the point $(x, y, \mathbf{z}) \in G_w$:

$$\begin{cases} \dfrac{\partial u^\delta}{\partial x} = \left(\dfrac{\sqrt{\delta^4 + r'^2}}{R'} \right) \dfrac{\partial u^\delta}{\partial x'}(x', y', \sqrt{\delta^4 + r'^2}) \\ \dfrac{\partial u^\delta}{\partial y} = \left(\dfrac{\sqrt{\delta^4 + r'^2}}{R'} \right) \dfrac{\partial u^\delta}{\partial y'}(x', y', \sqrt{\delta^4 + r'^2}) \\ \dfrac{\partial u^\delta}{\partial R} = \left(\dfrac{x'\sqrt{\delta^4 + r'^2}}{\delta^2 R'} \right) \dfrac{\partial u^\delta}{\partial x'}(x', y', \sqrt{\delta^4 + r'^2}) \\ \qquad\qquad + \left(\dfrac{y'\sqrt{\delta^4 + r'^2}}{\delta^2 R'} \right) \dfrac{\partial u^\delta}{\partial y'}(x', y', \sqrt{\delta^4 + r'^2}). \end{cases} \tag{51}$$

and in the same line by calculating the Jacobian of the new coordinates we have:

$$dx\, dy\, dR = \frac{\delta^2 R'^2}{(\delta^4 + r'^2)^{\frac{3}{2}}}\, dx' dy' dR'. \tag{52}$$

Using (29) and doing the same work, we get:

$$|\nabla u^\delta|^2 = \left| \frac{\partial u^\delta}{\partial x} \right|^2 + \left| \frac{\partial u^\delta}{\partial y} \right|^2 + \left| \frac{\partial u^\delta}{\partial R} \right|^2 + I \tag{53}$$

where

$$I = \frac{1}{R^2}\left(\left|\frac{\partial u^\delta}{\partial \theta_1}\right|^2 + \frac{1}{\sin^2\theta_1}\left|\frac{\partial u^\delta}{\partial \theta_2}\right|^2 + \cdots \right.$$
$$\left. + \frac{1}{\sin^2\theta_1 \sin^2\theta_2 \cdots sin^2\theta_{n-4}}\left|\frac{\partial u^\delta}{\partial \varphi}\right|^2\right).$$

Using (50) and applying (41) and (53) to the points of D_{δ^2} we obtain

$$\begin{aligned} I(x',y',R') &= \frac{(\delta-\delta^2)^2}{R^2} I(x',y',\sqrt{\delta^4+r'^2}) \\ &\le \frac{(\delta-\delta^2)^2}{R^2}\left|\nabla_{\mathbf{z}} u^\delta(x',y',\sqrt{\delta^4+r'^2})\right|^2 \\ &\le C\frac{(\delta-\delta^2)^2}{R^2} \end{aligned} \tag{54}$$

Therefore by integrating directly over the cone G_w we deduce from (54):

$$\int_{G_w} R^{n-3} I \, dx\, dy\, dR = O(\delta^{n+3}). \tag{55}$$

Furthermore considering (49), (51), (52) and (53) we estimate the integral

$$J = \int_{G_w} R^{n-3}(|\nabla u^\delta|^2 - I)$$

as follows

$$\begin{aligned} J &= \int_{G_w} R^{n-3}\left(\left|\frac{\partial u^\delta}{\partial x}\right|^2 + \left|\frac{\partial u^\delta}{\partial y}\right|^2 + \left|\frac{\partial u^\delta}{\partial R}\right|^2\right) dx\, dy\, dR \\ &= \int_{D_{\delta^2}} dx'\, dy' \int_0^{\sqrt{\delta^4+r'^2}} R^{n-3}\frac{\delta^2 R'^2}{(\delta^4+r'^2)^{\frac{3}{2}}} \\ &\quad \times \left[\frac{\delta^4+r'^2}{R'^2}\left|\nabla_{x'y'} u^\delta(x',y',\sqrt{\delta^4+r'^2})\right|^2 \right. \\ &\quad \left. + \left(\frac{\delta^4+r'^2}{\delta^4 R'^2}\right)\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right| + y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2\right] dR' \\ &= \int_{D_{\delta^2}} \frac{\delta^2}{\sqrt{\delta^4+r'^2}}\left|\nabla_{x'y'} u^\delta\right|^2 \int_{-\delta^2}^{\delta} \frac{\sqrt{\delta^4+r'^2}}{\delta^2} R^{n-3}\, dR \\ &\quad + \int_{D_{\delta^2}} dx'\, dy' \frac{1}{\delta^2\sqrt{\delta^4+r'^2}}\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right| + y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2 \\ &\quad \times \int_{\delta-\delta^2}^{\delta} \frac{\sqrt{\delta^4+r'^2}}{\delta^2} R^{n-3}\, dR. \end{aligned} \tag{56}$$

Using the inequality

$$|\nabla_{xy}u^\delta|^2 \le C\frac{\delta^8}{(\delta^8+r^2)^2} \quad \text{on} \quad D_{\delta^2} \tag{57}$$

which is established in [5] we obtain

$$\begin{aligned}\int_{D_{\delta^2}} dx'\,dy' \frac{1}{\delta^2\sqrt{\delta^4+r'^2}}\left(x'\left|\frac{\partial u^\delta}{\partial x'}\right| + y'\left|\frac{\partial u^\delta}{\partial y'}\right|\right)^2 \\ \times \int_{\delta-\delta^2}^{\delta} \frac{\sqrt{\delta^4+r'^2}}{\delta^2} R^{n-3}\,dR \\ \le C\int_0^{\delta^2} \delta^{n+3}\frac{r^3}{(\delta^8+r^2)^2}\,dr = O(\delta^{n+3}\ln(1/\delta)).\end{aligned} \tag{58}$$

And combining (38), (53), (55), (56) and (58), finally we get:

$$\begin{aligned}\int_{G_w} R^{n-3}|\nabla u^\delta|^2 = \frac{8\pi}{n-2}(\delta^{n-2}-(\delta-\delta^2)^{n-2}) \\ -\frac{8\pi}{n-2}c^2\delta^{n+2} + O(\delta^{n+3}\ln(1/\delta)).\end{aligned} \tag{59}$$

Now, using the estimates in (36) and the fact that $u^\delta = u$ on $\partial\tilde{c}^\delta\backslash\partial c^\delta$ we observe that $|\nabla u^\delta|$ is bounded on ∂H_w and therefore following the same method as the one used for G_w we get

$$\int_{H_w} R^{n-3}|\nabla u^\delta|^2 = O(\delta^{n+3}). \tag{60}$$

which conjugated with (48) and (59) yields

$$\begin{aligned}\int_{\tilde{c}^\delta} |\nabla u^\delta|^2 = 8\pi\omega_{n-2}(\delta^{n-2}-(\delta-\delta^2)^{n-2}) \\ -8\pi\omega_{n-2}c^2\delta^{n+2} + O(\delta^{n+3}\ln(1/\delta)).\end{aligned} \tag{61}$$

5.3.4 The estimate for the energy of u in C^δ

Similarly as in [14] we have the following estimate:

$$\begin{aligned}\int_{C^\delta} |\nabla u|^2 = 4\pi\omega_{n-2}\delta^{n+2}(a^2+b^2) \\ +4\pi\delta^4\int_{0\le R\le\delta-\delta^2} |\nabla_{\mathbf{z}} u(0,0,\mathbf{z})|^2 d\mathbf{z} + O(\delta^{n+3}).\end{aligned} \tag{62}$$

5.4 The end of proof of Lemma 4

Conjugating (37), (46) ,(61) and (62) we obtain:

$$\int_{\Omega} |\nabla u^{\delta}|^2 = 8\pi\omega_{n-2}\delta^{n-2}$$
$$-4\pi\omega_{n-2}\delta^{n+2}(4c^2 - (a^2 + b^2 + 8c^2 - 4ac - 4bc)\ln 2)$$
$$+O(\delta^{n+3}\ln(1/\delta)) \tag{63}$$

and by choosing a suitable c such that

$$4c^2 - (a^2 + b^2 + 8c^2 - 4ac - 4bc)\ln 2 > 0$$

we can be sure that for δ small enough $v = u^{\delta}$ would satisfy the strict inequality (32). For example put $c = \max\left\{\frac{a}{2}, \frac{b}{2}\right\}$. It is easy to verify that the degree of v on its only singular set, i.e. $\sigma = \{(0, 0, \mathbf{z})\,|R = \delta\}$ is one. By the way as in (17):

$$L(v, u) = \sup_{\substack{\psi \in \Omega^{\infty}_{n-3}(\overline{\Omega}) \\ |d\psi|_{\infty} \leq 1}} \left\{\int_{\Omega} v^*\omega \wedge d\psi - \int_{\Omega} u^*\omega \wedge d\psi\right\}$$
$$= \sup_{\substack{\psi \in \Omega^{\infty}_{n-3}(\overline{\Omega}) \\ |d\psi|_{\infty} \leq 1}} \mathbf{S}_v(\psi) \tag{64}$$

as $\mathbf{S}_u = 0$. Meanwhile using the corollary 1:

$$|\mathbf{S}_v(\psi)| = |\tau(\sigma, 1, \vec{\sigma})(\psi)| = |\mathbf{T}(d\psi)| \leq \mathbf{M}(\mathbf{T}) \tag{65}$$

for every current $\mathbf{T}$ which takes σ as its boundary, using the fact that $|d\psi|_{\infty} \leq 1$. Putting $\mathbf{T} = \mathbf{T}_0 = \tau(B_{\delta}, 1, \vec{B}_{\delta})$ where B_{δ} is the $(n-2)$-ball of the center 0 and of radius δ , we obtain combining (64) and (65):

$$L(v, u) \leq \omega_{n-2}\delta^{n-2}. \tag{66}$$

Now take $\psi_0 = z_1 \wedge dz_2 \wedge \cdots \wedge dz_{n-2}$. A simple observation shows that $\mathbf{T}_0(d\psi_0) = \mathbf{M}(\mathbf{T}_0) = \omega_{n-2}\delta^{n-2}$, so again using (64) and (65) we obtain easily that

$$L(v, u) \geq \omega_{n-2}\delta^{n-2}$$

which completes the proof regarding (66). □

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