PRESTRAINED ELASTICITY: FROM SHAPE FORMATION TO MONGE-AMPÈRE ANOMALIES

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1. Introduction

Imagine an airplane wing manufactured in a hyperbolic universe and imported into our Euclidean space. The incompatibility of the two geometries would be an obstacle for the relative ideal hyperbolic distances in the wing to be realized in the ambient Euclidean space. As a consequence, the wing would take on a deformed shape and be subject to internal stresses, making it not suitable for flying. This scenario, though imaginary, describes an everyday phenomenon known as *prestrain* in nonlinear elasticity. Here, prestrain refers to an incompatible ideal metric, and contrary to the above situation, it can play a positive role in nature and in applications.

Figure 1.1 shows the optimal 'relaxations' of a planar film allowed to freely seek a strain-minimizing deformation in space. Although the prescribed strain is radially symmetric, the resulting configurations are not; they exhibit large-scale buckling and multi-scale wrinkling, and in fact they still retain residual strain albeit smaller than the original one.
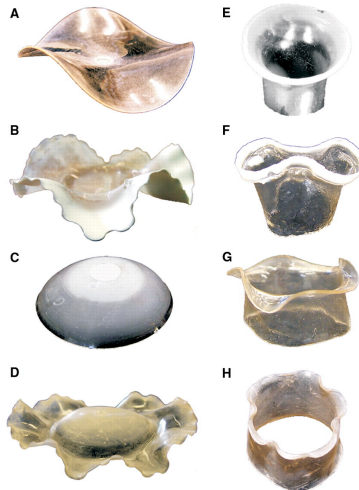

FIGURE 1.1. The minimizing shapes of thin films with radially symmetric strains (target metrics). Reprinted from Klein et al. [5] with permission from AAAS.

How 'good' are these relaxations in general? This problem can be studied through a variational model, pertaining to the non-Euclidean version of nonlinear elasticity, which postulates formation

of a target Riemannian metric resulting in the morphogenesis of the tissue that attains a configuration closest to being the metric's isometric immersion. It now turns out that the answer to the above question depends on the scaling of the energy minimizers in terms of the film's thickness and, a-posteriori, by the emerging isometry constraints on deformations with low regularity.

The study of mappings with weak regularity and the behavior of rough solutions to PDEs arising in geometry or physics has been an important part of analysis for decades. Many physical phenomena modeled by PDEs cannot be described by merely smooth solutions. On the other hand, lack of regularity can lead to nonphysical solutions, or even to situations where generically every function is close to a solution. This kind of mathematical behavior goes back to early work by Nash and Kuiper on isometric embeddings, where a Riemannian surface can be $\mathcal{C}^1$ isometrically embedded in $\mathbb{R}^3$, while higher smoothness requires higher dimensions.

In practical applications, thin films can be residually strained by a variety of means, such as: inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption, or opto-thermal stimuli in glass sheets. An interesting application, suggested by Kim et al. [4], creates curvy films by using light technology for the temperature-responsive flat gel sheets that transform into a prescribed curved surface when the in-built metric is activated.

We hope that the study of thin films will lead to a better understanding of three dimensional solids and such fundamentals as energy scaling laws, or the role of curvature and symmetry breaking. Current disagreements between theory and experiment need also to be resolved.

2. Incompatible elasticity and residual stresses

Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain and let G be a smooth Riemiannian metric on Ω. It is well known that when the Riemann curvature tensor $Riem(G)$ vanishes in Ω, there exists a mapping u (in other words, a *deformation*) of Ω into $\mathbb{R}^n$ which is an isometric immersion of G:

$$(2.1) \qquad\qquad \nabla u(x)^T \nabla u(x) = G(x) \quad \forall x \in \Omega.$$

When the mentioned condition fails (as it fails for a generic choice of G), one proceeds by seeking an orientation-preserving deformation u which minimizes the difference between the tensor fields in the right and the left hand sides of (2.1). This difference is measured by the energy functional, called the *prestrained (or incompatible) elasticity*:

$$(2.2) \qquad\qquad E(u) = \int_\Omega \mathrm{dist}^2\big(\nabla u(x) G(x)^{-1/2}, SO(n)\big) \ \mathrm{d}x$$

defined over the set of admissible deformations $u \in W^{1,2}(\Omega, \mathbb{R}^n)$ with square integrable derivatives of first order. The distance in matrix space $\mathbb{R}^{n \times n}$ is measured in terms of the Hilbert-Schmidt norm $\|A\|^2 = \mathrm{trace}(A^T A)$. Note that $E(u) = 0$ if and only if u is orientation preserving and satisfies (2.1). In this case, a change of variable reduces (2.2) to a standard nonlinear elasticity functional of the type: $\int_\Omega W(\nabla u) \ \mathrm{d}x$ which has been largely studied in the literature.

In the incompatible case when $Riem(G) \not\equiv 0$, existence of an energy gap phenomenon was shown in [8]. Namely, the equilibrium state of the body Ω must have a positive energy content: $\inf E > 0$, which we refer to as the *residual energy*. So far, only partial quantified estimates of this infimum in terms of $Riem(G)$ have been obtained. To better understand this problem, as well as to explore the relationship between the components of the target metric and the Riemann curvature as the driving force behind respectively the mechanical response and the residual stress, one is lead to study models with reduced complexity, e.g. through dimension reduction.

A thin film can be modeled by the Cartesian product $\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})$, with the *mid-plate* $\omega \subset \mathbb{R}^2$ and small thickness $h \ll 1$. In what follows, we are concerned with analyzing the infimum energy and the structure of minimizers of the energy functional below, now also in relation to the vanishing thickness $h \to 0$:

$$(2.3) \qquad E^h(u^h) = \frac{1}{h} \int_{\Omega^h} \mathrm{dist}^2\big((\nabla u^h)(G^h)^{-1/2}, SO(3)\big) \, \mathrm{d}x \qquad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3).$$

3. Γ-CONVERGENCE

A major difficulty in studying the functionals (2.3) is that the frame invariance of the energy density spoils convexity. Thus, in general, direct methods of calculus of variations cannot be applied and the minimizing sequences to (2.3) must be studied through asymptotic analysis, exploiting the small thickness of the domain. Namely, one first hopes to establish compactness properties for approximate minimizers of E^h as $h \to 0$. These, naturally, vary among different ranges of the scaling exponent β in: $\inf E^h \sim h^\beta$, which is in its turn induced by the prestrain G^h. Having found the admissible set of the limiting deformations one then looks for suitable 'dimensionally reduced' energies that would carry the structure of E^h. The method of Γ-*convergence* is one of the strategies available for this purpose in the variational toolbox.

In the present set-up for thin films, proving Γ-convergence of $h^{-\beta} E^h$ consists of deriving two inequalities. The first inequality establishes a lower bound: $\mathcal{I}_\beta(u) \le \liminf_{h \to 0} h^{-\beta} E^h(u^h)$, for any sequence u^h converging to a mapping u. The second inequality shows that the previous bound is optimal in the sense that for any given admissible u, we have $\mathcal{I}_\beta(u) = \limsup_{h \to 0} h^{-\beta} E^h(u^h)$ for a particular recovery sequence u^h converging to u.

The main feature of this definition, which in fact justifies its applicability, is that the limits of any converging sequence of minimizers of E^h coincide with the minimizers of $\mathcal{I}_\beta$. Again, the results vary and depend on the chosen scaling β; in general larger energies admit larger deformations, while smaller energies (induced by G^h with small Riemann curvatures in terms of h) admit only more restrictive deformations that need to preserve certain stringent curvature constraints.

4. CURVATURE DRIVEN ENERGY SCALING QUANTIZATION

We start by a short excursion in the context of compatible prestrains satisfying: $Riem(G) \equiv 0$. In this case, a change of variable brings the energy (2.3) to the standard nonlinear elasticity functional defined on deformations u^h of a tubular neighbourhood S^h of a surface $S \subset \mathbb{R}^3$, with trivial prestrain $G = \mathrm{Id}_3$. When $S = \omega \subset \mathbb{R}^2$, the quantitative geometric rigidity estimate established in [3] lead to the rigorous study of the dimensionally reduced thin models in low energy scalings. For more general geometries, a conjecture has been put forward [9] concerning an infinite hierarchy of limiting thin shell models, each valid in its respective energy scaling regime induced by the scaling of the applied body forces. In each case, the Γ-limit of $h^{-\beta} E^h$ consisted of a computable combination of bending and stretching.

In certain situations, the geometry of S allows for the matching of a lower order infinitesimal isometries to higher order ones, whereas the corresponding theories collapse to one and the same theory, valid under the lower order infinitesimal isometry constraint. The conjecture and this '*collapse phenomenon*' is so far consistent with all the rigorously established analytical results.

The picture in the prestrained elasticity scenario, where $Riem(G) \not\equiv 0$, is richer in as much as it does not generate one sequential hierarchy but rather a network of limiting models, differentiated by the scaling of the components of the curvatures of G^h when $h \to 0$.

When $G^h = G$ is independent of thickness parameter, an energy gap phenomenon can be observed [1]. Namely, the only possible scaling after the non-zero energy drops below h^2, is that of order h^4. In the first case, the Γ-limit of $h^{-2}E^h$ consists of a curvature functional defined over the $W^{2,2}$ isometric immersions of the two dimensional manifold $(\omega, G_{2\times 2})$ into $\mathbb{R}^3$. In the second case, the three Riemann curvatures R_{1212}, R_{1213} and R_{1223} of G vanish identically. The Γ-limit of $h^{-4}E^h$ is then given in terms of *stretching* i.e. the change of metric, and *bending* that is the induced change of the second fundamental form, with respect to the unique isometric immersion that gives the zero energy in the prior Γ-limit; plus a new term, that quantifies exactly the remaining three possibly non-zero Riemann curvatures.

5. The Monge-Ampère constrained energy

The Monge-Ampère equation:

$$(5.1) \qquad\qquad \det \nabla^2 v = f \quad \text{in } \omega \subset \mathbb{R}^2,$$

can be seen as a 'small slope' variant of the isometric immersion equations and it naturally arises in the thin limit residual theories of the model (2.3). Indeed, for the incompatibility tensor of the form $G^h = \mathrm{Id}_3 + 2h^\gamma S$ where $0 < \gamma < 2$, the Γ-limit $\mathcal{I}$ of $h^{-(\gamma+2)}E^h$ is effectively defined [7, 6] on the deformations of regularity $W^{2,2}$ for whom the pull-back of the Euclidean metric coincides with the prestrain G^h at the first order of expansion of their Gauss curvatures. This condition is precisely equivalent to (5.1) with $f = -\mathrm{curl}^T \mathrm{curl}\, S_{2\times 2}$, whereas we have: $\mathcal{I}(v) = \int_\omega |\nabla^2 v|^2$.

For future purpose, let us note that the above discussion motivates the following weak form of the two dimensional Monge-Ampère equation (5.1):

$$(5.2) \qquad\qquad \boxed{\mathcal{D}et\nabla^2 v := -\tfrac{1}{2}\mathrm{curl}^T\mathrm{curl}(\nabla v \otimes \nabla v)} = f.$$

The Monge-Ampère constrained variational problem $\mathcal{I}$ is the source of a wide range of questions; from the technical obstacles in deriving the model as a Γ-limit, to the study of regularity and multiplicity of minimizers or critical points, of which many remain open. In this line, we recently demonstrated the surprising existence of a class of anomalous solutions to (5.2). The rest of this article is dedicated to this line of inquiry.

6. Convex integration for the Monge-Ampère equation

When f is non-negative, any $v \in W^{2,2}(\omega)$ satisfying (5.1) must actually be $\mathcal{C}^1$ and *convex*. Once the convexity is established, the path opens up for applying the standard results in the theory of nonlinear PDEs to obtain better interior regularity of v depending on the given regularity of f. For the 'flat case' $f \equiv 0$, any such v must be *developable*: it is $\mathcal{C}^1$ and for every point $x \in \omega$ there exists either a neighborhood of x, or a segment passing through it and joining $\partial\omega$ at its both ends, on which ∇v is constant.

The same assertions of convexity/developability are true [10] for solutions $v \in \mathcal{C}^{1,\alpha}(\omega)$ of (5.2) with $\frac{2}{3} < \alpha < 1$. Let us point out that a crucial step in proving results for the weak Hölder regular solutions, is a commutator estimate which yields a degree formula for the Hölder continuous mapping ∇v. Such commutator estimates were used for the Euler equations by Constantin, E and Titi, and for the isometric immersion problem by Conti, Delellis and Szekelyhidi; this relationship is not surprising in view of the presence of a quadratic term in the equations in all three cases.

The parallels with the isometric immersions and Euler's equations do not stop here. In both cases, the known *rigidity* statements are contrasted with existence of anomalous *flexible* solutions

in lower regular regimes. It is perhaps surprising that similar statements on existence of anomalous solutions to the Monge-Ampère equation (5.1) have been missing in the literature. Indeed, the reformulation (5.2) leads to the following counter intuitive result [10]. Fixing an exponent $\alpha < \frac{1}{7}$ and the right hand side $f \in L^{7/6}(\omega)$, the set of $\mathcal{C}^{1,\alpha}(\bar{\omega})$ solutions to (5.2) is dense in $\mathcal{C}^0(\bar{\omega})$.

The critical value of Hölder's exponent at the threshold of rigidity and flexibility is not yet clear; it has been conjectured to be $\frac{1}{3}, \frac{1}{2}$ or $\frac{2}{3}$, relying on various intuitions. Here, and also in the case of isometries, the Nash-Kuiper iteration method cannot yield anomalous solutions with regularity better than $\mathcal{C}^{1,1/3}$, but on the other hand there seem to be little indication of how to prove the rigidity for the regimes $\frac{1}{3} < \alpha \le \frac{2}{3}$. This situation is, again, parallel with the recent results in the context of fluid dynamics (see Delellis and Szekelyhidi [2] and the references therein) where the famous Onsager's conjecture puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at exactly $\mathcal{C}^{0,1/3}$.

7. Conclusion

In this article, we motivated how the prestrain metric problem can be formulated for three dimensional elastic bodies and showed how it leads to problems in geometry and analysis. In particular, rigidity properties of the weak solutions to geometric PDEs come to the frontline, including the discovery of the anomalous solutions to the Monge-Ampère equation. The investigation of the dimensionally reduced models can also shed light on the precise role which is played by the curvature tensor in the stress distribution within a three dimensional body and eventually lead to a better understanding of the shape formation phenomena through growth, plasticity, etc.

Coming back to the energy (2.3), a direct consequence of the existence of the anomalous $\mathcal{C}^{1,\alpha}$ solutions in the regime $\alpha < 1/7$, is that for all given $G^h = \mathrm{Id}_3 + 2h^\gamma S$ one has: $\inf E^h \ll h^{1/2}$. This could be improved to: $\inf E^h \ll h$, if the anomalous regime was extended to $\alpha < 1/3$. Finally, scaling regimes between h^2 and $h^{1/2}$, and the corresponding behaviour of thin prestrained films, are not yet well understood.

Other largely unexplored related topics include homogenization, symmetry and symmetry breaking, inverse prestrain analysis (useful e.g. in tumor detection) and randomly generated prestrain. These avenues of research connect between theory of elasticity, differential geometry, analysis and PDEs. W also hope that a thorough theoretical understanding of the phenomena discussed in this article could help in engineering sheets or bodies with finely controlled shapes, dynamics, structural resistance to loads and elastic properties such as rigidity and flexibility.

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