

PRESTRAINED ELASTICITY: FROM SHAPE FORMATION TO MONGE-AMPÈRE ANOMALIES

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1. INTRODUCTION

This article is concerned with the analytical and geometrical questions emerging from the study of thin elastic films that exhibit residual stress at free equilibria. Examples of such structures and their actuations are present in many applications and they include: plastically strained sheets; specifically engineered swelling or shrinking gels; growing tissues; atomically thin graphene layers, etc. These and other phenomena can be studied through a variational model, pertaining to the non-Euclidean version of nonlinear elasticity, which postulates formation of a target Riemannian metric, resulting in the morphogenesis (i.e. shape formation) of the tissue that attains an orientation-preserving configuration closest to being the metric's isometric immersion.

In this context, analysis of scaling of the energy minimizers in terms of the film's thickness leads to the rigorous derivation of a hierarchy of limiting variational theories, differentiated by the embeddability properties of the target metrics and, a-posteriori, by the emergence of isometry constraints on deformations with low regularity. Many problems regarding multiplicity, singularities and regularity of the critical points of the derived models remain open and are hard to tackle due to, generally speaking, lack of convexity. On the other hand, these problems lead to further questions of rigidity and flexibility of solutions to the weak formulations of the related PDEs, including the weak Sobolev or Hölder solutions to the Monge-Ampère equation. One particular result in this line is that the set of $C^{1,\alpha}$ solutions to the Monge-Ampère equation in two dimensions is dense in C^0 provided that $\alpha < 1/7$, whereas rigidity holds when $\alpha > 2/3$.

The discussion of thin limit models as above, has consequences for the three dimensional original model in terms of energy scaling laws, understanding of the role of curvature in determining the mechanical properties of the material, and in the effects of the symmetry and the symmetry breaking solutions. There are still unresolved dichotomies between theory and experiments which call for a thorough understanding of the above phenomena in their proper geometrical and analytical contexts.

2. INCOMPATIBLE ELASTICITY AND RESIDUAL STRESSES

Let $\Omega \subset \mathbb{R}^n$ be a simply connected domain and let $G : \Omega \rightarrow \mathbb{R}_{sym,pos}^{n \times n}$ be a smooth Riemannian metric on Ω . It is well known that the manifold (Ω, G) can be isometrically immersed in \mathbb{R}^n if and only if its Riemann curvature tensor vanishes in Ω , i.e: $Riem(G) \equiv 0$. When this condition holds, there exists a smooth mapping u of Ω into \mathbb{R}^n , called in what follows a *deformation*, that is an isometric embedding of G :

$$(2.1) \quad \nabla u(x)^T \nabla u(x) = G(x) \quad \forall x \in \Omega.$$

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When the mentioned condition fails (as it fails for a generic choice of G), one proceeds by seeking an orientation-preserving deformation u that minimizes the difference between the tensor fields in the right and the left hand sides of (2.1). This is done by postulating a variational model called the *prestrained (or incompatible) elasticity*; the objective is now to study the critical points, minimizers or almost minimizers of the energy functional:

$$(2.2) \quad E(u) = \int_{\Omega} \text{dist}^2(\nabla u(x)G(x)^{-1/2}, SO(n)) \, dx$$

defined over the set of admissible vector fields with square integrable derivatives of first order, i.e.: $u \in W^{1,2}(\Omega, \mathbb{R}^n)$. Above, $SO(n)$ stands for the special orthogonal group of proper rotations, while the distance of a matrix B (here, B is the product of two matrices $\nabla u(x)$ and $G(x)^{-1/2}$ at each $x \in \Omega$) from the compact set $SO(n)$ in $\mathbb{R}^{n \times n}$ is simply the minimal distance $|B - R| = (\text{Trace}(B - R)^T(B - R))^{1/2}$ from all elements $R \in SO(n)$.

More generally, for a description of an elastic prestrained material with reference configuration $\Omega \subset \mathbb{R}^3$, one also considers the energy:

$$(2.3) \quad E_W(u) = \int_{\Omega} W(\nabla u(x)G(x)^{-1/2}) \, dx.$$

The density function $W : \mathbb{R}^{3 \times 3} \rightarrow [0, \infty]$ is assumed to satisfy the physical conditions of: (i) frame invariance: $W(RB) = W(B)$ for all $B \in \mathbb{R}^{3 \times 3}$ and $R \in SO(3)$; and (ii) normalisation: $W(\text{Id}_3) = 0$. Consequently, both W and its first derivative DW vanish on the energy well $SO(3)$. To fix the ideas, we further assume that W has a quadratic growth away from $SO(3)$ i.e.: $W(B) \geq c \, \text{dist}^2(B, SO(3))$ with a uniform positive constant c .

Note that $E(u) = 0$ (or $E_W(u) = 0$) if and only if u is a smooth, orientation preserving isometric embedding of (Ω, G) into \mathbb{R}^n as in (2.1). In this case, a change of variable reduces (2.2) to a standard nonlinear elasticity functional: $\int_{\Omega} W(\nabla u) \, dx$ which has been largely studied in the literature (see [1, 3] and references therein). On the other hand, in the incompatible case when $\text{Riem}(G) \neq 0$, existence of an energy gap phenomenon $\inf E > 0$ was shown in [20]. In other words, the equilibrium state of the elastic body has positive energy content, which we refer to as the *residual prestrain energy*. So far, only partial quantified estimates of this infimum in terms of $\text{Riem}(G)$ have been obtained in [15]. To better understand this problem as well as to explore the relationship between the components of the target metric and the Riemann curvature tensor as the driving force behind respectively the mechanical response and the residual stress, one is lead to study models with reduced complexity, e.g. through dimension reduction.

3. THIN FILMS AND DIMENSION REDUCTION IN PRESTRAINED ELASTICITY

In practical applications, thin films can be residually strained by a variety of means such as: inhomogeneous growth, plastic deformation, swelling or shrinkage driven by solvent absorption, or opto-thermal stimuli in liquid nematic glass sheets. An instructive effort to reproduce the effect of the prestrain on the shape of thin films in an artificial setting was reported in [13]. There, thin gel films were manufactured with the property that they underwent nonuniform shrinkage when activated in a hot bath (see Figure 3.1) and hence realised a prescribed prestrain on the sheets. Both large-scale buckling and multi-scale wrinkling structures appeared in the sheets, depending on the nature of the prescribed metrics. More recently, another approach to controlling the shape through prestrain was suggested in [12], through a method of photopatterning polymer films (see Figure 3.2), yielding the temperature-responsive flat gel sheets that transform into a prescribed

curved surface when the in-built metric is activated. For other experimental results see for example [25, 29].

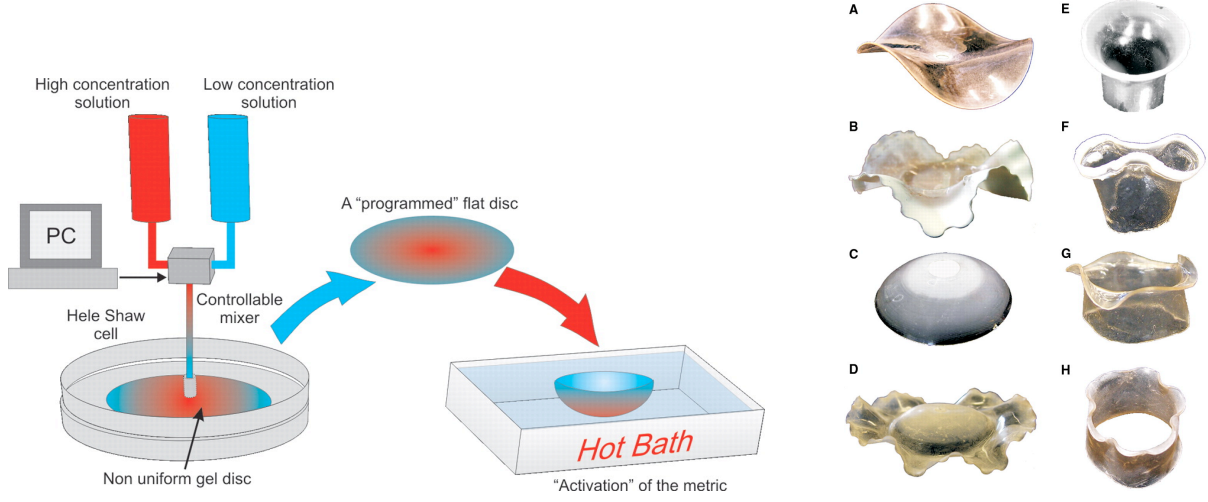


FIGURE 3.1. The experimental system and the obtained structures of sheets with radially symmetric target metrics. Reprinted from [13] with permission from AAAS.

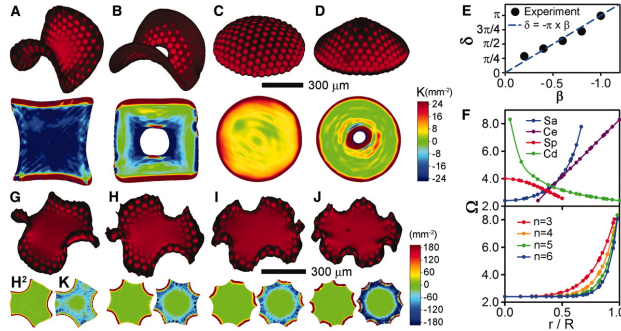


FIGURE 3.2. Halftone gel lithography from Kim et al. [12]: shapes obtained by photopatterning polymer films. Reprinted with permission from AAAS..

A thin film can be modeled by the Cartesian product $\Omega^h = \omega \times (-\frac{h}{2}, \frac{h}{2})$, with the *mid-plate* $\omega \subset \mathbb{R}^2$ and small thickness $h \ll 1$. As in (2.3), we are concerned with analyzing the infimum energy and the structure of minimizers of the energy functional below, now also in relation to the vanishing thickness $h \rightarrow 0$:

$$(3.1) \quad E^h(u^h) = \frac{1}{h} \int_{\Omega^h} W((\nabla u^h)(G^h)^{-1/2}) \, dx \quad \forall u^h \in W^{1,2}(\Omega^h, \mathbb{R}^3).$$

For a general description, we assume that the incompatibility tensors are determined via a family of metrics G^h . Below, we will showcase two distinct situations: the first one when $G^h = G$ that is constant along the thickness, and the second one when G^h are a thickness-dependent perturbations of the trivially immersable (hence inducing stress-free equilibria) metric Id_3 .

3.1. Γ -convergence. A major difficulty in studying the functional (3.1) is that the frame invariance of W contradicts the possibility of imposing suitable convexity assumptions. Thus, in general, direct methods of calculus of variations cannot be applied and the minimizing sequences of the family of problems (3.1) are studied through asymptotic analysis, exploiting the small thickness of the domain. Namely, one first hopes to establish compactness properties for sequences of approximate minimizers of E^h as $h \rightarrow 0$. These will naturally vary among different ranges of the scaling exponent β in: $\inf E^h \sim h^\beta$, that is in its turn induced by the prestrain encoded in the curvatures of G^h . Having found the admissible set of the limiting deformations $u \in \mathcal{A}_\beta$, one then looks for suitable “dimensionally reduced” energies defined on \mathcal{A}_β that would carry the structure of E^h . The method of Γ -convergence is one of the strategies available for this purpose in the variational toolbox.

In the present set-up for thin films, proving Γ -convergence of the scaled energies $\frac{1}{h^\beta} E^h$ consists of deriving two inequalities. Fix a metric topology on the space of deformations u^h . The first inequality establishes a lower bound: $\mathcal{I}_\beta(u) \geq \liminf_{h \rightarrow 0} \frac{1}{h^\beta} E^h(u^h)$, for any sequence u^h converging to a mapping u . The second inequality shows that the previous bound is optimal in the sense that for any given admissible $u \in \mathcal{A}_\beta$, we have $\mathcal{I}_\beta(u) = \limsup_{h \rightarrow 0} \frac{1}{h^\beta} E^h(u^h)$ for a particular *recovery sequence* u^h converging to u . We then say that $\frac{1}{h^\beta} E^h$ Γ -converges to the residual energy \mathcal{I}_β .

The main feature of the above definition, which in fact justifies its applicability, is that it implies immediately that the limits of any converging sequence of minimizers (or approximate minimizers) of E^h coincide with the minimizers of \mathcal{I}_β , hence identifying the governing variational principle for the asymptotic behavior of the possible minimizers u^h of (3.1). Applying this set-up needs specific understanding of the problem at hand and also the usage of various analytical and geometric techniques. As expected, the results vary and depend on the chosen scaling of the energies or the prestrain metrics; in general larger energies admit larger deformations, while smaller energies (induced by the metrics G^h with small Riemann curvatures in terms of the parameter h) admit only more restrictive deformations, that need to preserve certain stringent curvature constraints or that only slightly depart from a trivial isometric immersion. All done, it still remains to investigate the properties of the minimizers in the set \mathcal{A}_β which despite their better accessibility, pose another challenging task due to the presence of non-convex curvature or isometry type constraints.

3.2. Showcase I: Curvature driven energy scaling quantisation. In papers [20, 2, 23] the low energy limit theories have been discussed in case when the prestrain $G^h = G$ is constant along the thickness, i.e. $G(x) = G(x')$, where we denote $x = (x', x_3)$. We will denote by $G_{2 \times 2}$ the principal minor of the tensor field G , that is a metric on the two dimensional midplate ω .

Firstly, the critical energy scaling: $\inf E^h \sim h^2$ takes place if and only if the following two conditions are simultaneously satisfied: (a) There exists a $W^{2,2}$ isometric immersion of the two dimensional manifold $(\omega, G_{2 \times 2})$ into \mathbb{R}^3 , i.e. the residual set \mathcal{A} below is nonempty. (b) At least one of the three Riemann curvatures R_{1212}, R_{1213} or R_{1223} of G does not vanish identically. Secondly, the scaled energies $\frac{1}{h^2} E^h$ Γ -converge to the following curvature (Kirchhoff-like) functional:

$$(3.2) \quad \mathcal{I}_2(y) = \frac{1}{24} \int_{\omega} \mathcal{Q}_2(x', (\nabla y)^T \nabla \vec{b}) dx',$$

defined on the set of admissible deformations y of ω , that are the isometric immersions of the midplate metric $G_{2 \times 2}$ with two derivatives square integrable:

$$(3.3) \quad \mathcal{A}_2 = \{y \in W^{2,2}(\omega, \mathbb{R}^3); (\nabla y)^T \nabla y = G_{2 \times 2}\}.$$

The density $\mathcal{Q}_2(x', \cdot)$ in (3.2) is given by nonnegative quadratic forms defined explicitly [2] in terms of W , while $\vec{b} \in W^{1,2} \cap L^\infty(\Omega, \mathbb{R}^3)$ is the Cosserat vector field, uniquely given by requesting:

$$(3.4) \quad Q := [\partial_1 y, \partial_2 y, \vec{b}] \quad \text{satisfies} \quad Q^T Q = G \quad \text{and} \quad \det Q > 0.$$

In other words, \vec{b} determines the preferred direction of “stacking” copies of surfaces $y(\omega)$ on top of each other, in order to obtain the deformed three dimensional shell $u^h(\Omega^h)$ with minimal energy in (3.1). Note that if one could write $\vec{b} = \partial_3 y$, then (3.4) would say precisely that the induced three dimensional extension of y is an orientation preserving isometric immersion of G . For incompatible G , such immersion does not exist, yet still the optimal residual deformation y is singled out by minimizing its bending content (3.2) with respect to the curvature form $(\nabla y)^T \nabla \vec{b}$. This form reduces to the second fundamental form of the surface $y(\omega)$ in case \vec{b} happens to be the normal vector, but in general \vec{b} contains also the shear directions.

Thirdly, an energy gap phenomenon can be observed. The only possible scaling after the non-zero energy drops below h^2 , is that of order: $\inf E^h \sim h^4$. In this case, the Γ -limit of $\frac{1}{h^4} E^h$ is given in terms of the infinitesimal isometries and admissible strains on the surface isometrically immersing $G_{2 \times 2}$, with an extra curvature term:

$$\begin{aligned} \mathcal{I}_4(V, \mathbb{S}) &= \frac{1}{2} \int_{\omega} \mathcal{Q}_2(x', \text{stretching of order } h^2) \, dx' + \frac{1}{24} \int_{\omega} \mathcal{Q}_2(x', \text{bending of order } h) \, dx' \\ &\quad + \frac{1}{1440} \int_{\omega} \mathcal{Q}_2(x', \left[\begin{array}{cc} R_{1313} & R_{1323} \\ R_{1323} & R_{2323} \end{array} \right]) \, dx'. \end{aligned}$$

The functional \mathcal{I}_4 is the von Kármán-like energy, consisting of *stretching* i.e. the change of metric, and *bending* that is the induced change of the second fundamental form, with respect to the unique isometric immersion that gives the zero energy in the prior Γ -limit (3.2); plus a new term, that quantifies exactly the remaining three possibly non-zero Riemann curvatures.

3.3. Compatible prestrains and hierarchy of theories with isometry constraints.

We now make a short excursion in the context of compatible prestrains satisfying: $Riem(G) \equiv 0$. In this case, a change of variable brings the energy (3.1) to the standard nonlinear elasticity functional form: $E^h(u^h) = \frac{1}{h} \int_{S^h} W(\nabla u^h)$, defined on deformations u^h of a tubular neighbourhood S^h of a surface $S \subset \mathbb{R}^3$, but with the prestrain reduced to $G = \text{Id}_3$.

When $S = \omega \subset \mathbb{R}^2$, the quantitative geometric rigidity estimate established in [8] lead to the rigorous study of the dimensionally reduced thin models in low energy scalings. For more general geometries, in [21] a conjecture has been put forward concerning an infinite hierarchy of limiting thin shell models, each valid in its respective energy scaling regime induced by the scaling of the applied body forces. One can see through formal calculations, that if $E^h(u^h) \sim h^\beta$ and $\beta \in [\beta_k, \beta_{k+1})$, where the critical exponents are given by: $\beta_k = 2 + 2/k$, then u^h is asymptotically an infinitesimal isometry of order k on S . Namely, writing $\epsilon = h^{\beta/2-1}$ one has:

$$(3.5) \quad (u^h)|_S \sim \phi_\epsilon = \text{id} + \epsilon V_1 + \dots + \epsilon^k V_k \quad \text{and} \quad (\nabla \phi_\epsilon)^T \nabla \phi_\epsilon - \text{Id}_2 = o(\epsilon^k).$$

In other words, the one-parameter family of deformations ϕ_ϵ of S induce the change of metric whose $(k+1)$ th order terms in ϵ disappear.

The Γ -limit of $h^{-\beta} E^h$ in the above scenario consists of bending of order ϵ ; plus stretching of the lowest non-zero order ϵ^{k+1} in case $\beta = \beta_k$.

In certain situations, the geometry of S allows for the matching of a lower order infinitesimal isometries to higher order ones, whereas the corresponding theories over the specific range $[\beta_k, \beta_m]$

collapse to one and the same theory valid under the lower order infinitesimal isometry constraint. The conjecture and this “*collapse phenomenon*” is so far consistent with all the rigorously established analytical results. One interesting case scenario is that of convex S , where in [18] already the 1st order infinitesimal isometry constraint was shown to be a sufficient condition for matching to a full isometry. Consequently, the whole hierarchy of theories for scalings smaller than h^2 collapse into a single “linear” elasticity theory.

Another interesting case is that of plates ($S = \omega \subset \mathbb{R}^2$), where in [8] the 2nd order isometry constraint was shown to imply matching to a full isometry. Therefore, all scalings larger than h^4 and smaller than h^2 induce the same limiting theory, consisting of minimizing a biharmonic energy functional with a 2nd order isometry constraint, which in its turn translates into a degenerate Monge-Ampère equation on the out-of-plane displacement $v : \omega \rightarrow \mathbb{R}$. Indeed, rewriting the first expansion condition in (3.5) as:

$$(3.6) \quad \phi_\varepsilon = id + \varepsilon v e_3 + \varepsilon^2 w : \omega \rightarrow \mathbb{R}^3,$$

it is easy to check that the second condition in (3.5) holds if and only if $\text{sym} \nabla w = -\frac{1}{2} \nabla v \otimes \nabla v$. Further, existence of w with this property is equivalent to:

$$(3.7) \quad \det \nabla^2 v = 0 \quad \text{in } \omega.$$

The picture in the prestrained elasticity scenario, where $\text{Riem}(G) \not\equiv 0$, is richer in as much as it does not generate one sequential hierarchy but rather a network of limiting models, differentiated by the scaling of the components of the curvatures tensor for G^h when $h \rightarrow 0$. In [16, 19] a few of these models are discussed. In particular, the mentioned Monge-Ampère constraint arises again as a source of many questions regarding the rigidity, regularity and approximation properties of Sobolev or Hölder weakly regular solutions. In particular, we are lead to the surprising existence of a class of anomalous solutions to the Monge-Ampère equation (formulated through a very weak Hessian determinant operator) in two dimensions. The rest of this article is dedicated to this line of inquiry inspired by the thin prestrained model energy derived in [19].

3.4. Showcase II: The Monge-Ampère constrained energy. The Monge-Ampère equation can be seen as a “small slope” variant of the isometric immersion equation (2.1) and it naturally arises in the thin limit residual theories of the model (3.1). In [19, 17] the following incompatibility tensor was considered:

$$(3.8) \quad (G^h)^{1/2}(x', x_3) = \text{Id}_3 + h^\gamma S(x') + h^{\gamma/2} x_3 B(x'),$$

where $S, B : \bar{\omega} \rightarrow \mathbb{R}^{3 \times 3}$ are two given smooth matrix fields, and the scaling exponent γ belongs to the range $0 < \gamma < 2$. The residual theory is then effectively defined on the limiting deformations of regularity $W^{2,2}$ for whom the pull-back of the Euclidean metric coincides with the prestrain G^h only at the first order of expansion of their Gauss curvatures. Namely, $\frac{1}{h^{\gamma+2}} E^h$ Γ -converge to:

$$(3.9) \quad \mathcal{I}_f(v) = \frac{1}{12} \int_{\omega} \mathcal{Q}_2(\nabla^2 v + (\text{sym } B)_{2 \times 2})$$

defined on the set of the admissible out-of-plane displacements:

$$(3.10) \quad \mathcal{A}_f = \{v \in W^{2,2}(\omega, \mathbb{R}); \det \nabla^2 v = f\} \quad \text{where } f = -\text{curl}^T \text{curl } S_{2 \times 2}.$$

Further: $\inf E^h \sim h^{\gamma+2}$ if and only if the following two conditions are simultaneously satisfied: (a) The set \mathcal{A}_f is nonempty, (b) $\text{curl}(\text{sym } B)_{2 \times 2} \not\equiv 0$ or $\text{curl}^T \text{curl } S_{2 \times 2} + \det(\text{sym } B)_{2 \times 2} \not\equiv 0$.

The proofs rely on fine properties of the Sobolev solutions to the Monge-Ampère equation:

$$(3.11) \quad \det \nabla^2 v = f,$$

studied in the now classical unpublished preprint [28] in the $W^{2,\infty}$ regularity case. In [17] the same techniques were shown to work in $W^{2,2}$ setting, regularity being proved for positive curvature g .

In order to better understand the constraint in (3.10), we point out a connection between its solutions and the isometric immersions of Riemannian metrics, motivated by our studies of nonlinear elastic plates. Since on a simply connected domain ω , the kernel of the differential operator $\text{curl}^T \text{curl}$ consists of the fields of the form $\text{sym} \nabla w$, a solution to (3.7) can be characterized by the criterion:

$$(3.12) \quad \exists w : \Omega \rightarrow \mathbb{R}^2 \quad \frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w = 0 \quad \text{in } \omega.$$

As noted in the previous section, equation in (3.12) is an equivalent condition for the family (3.6) to form a 2nd order infinitesimal isometry. Since here we are dealing with isometries relative to an incompatible metric G , (3.12) is replaced by the equality of the tensor $T(v, w) = \frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w$ with a matrix field $S_{2 \times 2}$ that satisfies: $-\text{curl}^T \text{curl} S_{2 \times 2} = f$:

$$(3.13) \quad T(v, w) = S_{2 \times 2}.$$

Clearly, there are many potential choices for $S_{2 \times 2}$, for example one may take $S_{2 \times 2}(x') = \lambda(x') \text{Id}_2$ with $\Delta \lambda = -f$ in ω . Again, equation (3.13) states precisely that the metric $(\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon$ given by ϕ_ε in (3.6) agrees with the given metric $G_{2 \times 2} = \text{Id}_2 + 2\varepsilon^2 S_{2 \times 2}$ on ω , up to terms of order ε^2 . The Gauss curvature κ of $G_{2 \times 2}$ satisfies:

$$\kappa(G_{2 \times 2}) = \kappa(\text{Id}_2 + 2\varepsilon^2 S_{2 \times 2}) = -\varepsilon^2 \text{curl}^T \text{curl} S_{2 \times 2} + o(\varepsilon^2),$$

while $\kappa((\nabla \phi_\varepsilon)^T \nabla \phi_\varepsilon) = -\varepsilon^2 \text{curl}^T \text{curl}(\frac{1}{2} \nabla v \otimes \nabla v + \text{sym} \nabla w) + o(\varepsilon^2)$, so the problem (3.11) can also be interpreted as seeking for all appropriately regular out-of-plane displacements v that can be matched, by a higher order in-plane displacement perturbation w , to achieve the prescribed Gauss curvature f of ω , at its highest order term.

The Monge-Ampère constrained variational problem (3.9) (3.10) is the source of a wide range of challenging problems in analysis of Sobolev solutions to (3.11), ranging from the technical obstacles in deriving the model as a Γ -limit, to the study of regularity and multiplicity of minimizers or critical points. Many questions remain open, even in the positive curvature case $f \geq c > 0$, yielding the ellipticity of the constraint operator inside the domain. The following is a prototypical challenging open problem in this context: Assume that f is a positive smooth function on $\bar{\omega}$. Are smooth functions dense in \mathcal{A}_f with respect to the $W^{2,2}$ topology?

4. RIGIDITY FOR THE MONGE-AMPÈRE EQUATION

Motivated by the above discussion of the variational model (3.9) (3.10), one is led to study the two dimensional Monge-Ampère equation (3.11) written in the weak form:

$$(4.1) \quad \mathcal{D}et \nabla^2 v := -\frac{1}{2} \text{curl}^T \text{curl}(\nabla v \otimes \nabla v) = f.$$

Note that $W^{2,2}$ solutions of (4.1) coincide with the solutions of the Monge-Ampère equation (3.11).

Straightforward analytical observations about \mathcal{C}^2 regular solutions to (3.11) imply that they are convex (modulo a change of sign) if $f \geq c > 0$ and that they are *developable* if $f \equiv 0$. A clear

statement of rigidity for (3.11) is still lacking for the general f , as is the case for isometric immersions, where rigidity results are usually formulated only for the strictly elliptic [4] or Euclidean metrics [26, 11]. These two corresponding cases we will discuss below.

4.1. The positive curvature case $f \geq c > 0$. In case f is non-negative, a solution to (4.1) is called rigid if it is convex modulo a change of sign. In [28, 17] it was proved that any $v \in W^{2,2}(\omega)$ satisfying (3.11) in this case, must actually be \mathcal{C}^1 regular and locally convex. Moreover, v is then an Alexandrov solution to (3.11). Note that once the convexity is established, the path opens up for applying the standard results in the theory of nonlinear PDEs to obtain better interior regularity of v depending on the given regularity of f .

Further, in [22] we showed that any $v \in \mathcal{C}^{1,\alpha}(\omega)$ that is a solution to (4.1) with $\frac{2}{3} < \alpha < 1$ for f being a positive Dini continuous function, must be convex. In fact, v is also an Alexandrov solution, as before.

4.2. The degenerate case $f \equiv 0$. The “flat case” turns out to be more complicated. In [26] it has been shown that any $v \in W^{2,2}(\omega)$ that solves (3.7) must satisfy $v \in \mathcal{C}^1(\omega)$; and for every point $x \in \omega$ there exists either a neighborhood of x , or a segment passing through it and joining $\partial\omega$ at its both ends, on which ∇v is constant.

The $W^{2,2}$ hypothesis of [26] is optimal. Indeed, conic solutions to (3.7) exist if the regularity is assumed to be only $W^{2,p}$ for $p < 2$. One could even construct more sophisticated solutions by gluing these conic singularities in a suitable manner, using Vitali’s covering theorem, or establish the existence of strictly convex $W^{2,p}$ solutions to the more sophisticated equation (4.2) below when $p < 2$. In the meantime, it is known that for $p < 2$, there exist $W^{2,p}$ solutions to (4.1) which are not \mathcal{C}^1 and which fail to satisfy the developability statement at a given point in the domain.

A more general result in [10] applies to the (larger) class of Monge-Ampère functions. Namely, the regularity and developability conclusions above hold true for $v \in W^{2,1}(\omega)$ satisfying:

$$(4.2) \quad \int_{\omega} \phi_{x_1}(x, \nabla v) v_{x_k x_2} - \phi_{x_2}(x, \nabla v) v_{x_k x_1} \, dx = 0 \quad \forall \phi \in \mathcal{C}_c^{\infty}(\omega \times \mathbb{R}^2) \quad \forall k = 1, 2.$$

Finally, in [22] it has been proved that any $v \in \mathcal{C}^{1,\alpha}(\omega)$ solving (4.1) with $f = 0$, must be developable as long as $\frac{2}{3} < \alpha < 1$.

It is noteworthy that a crucial step in proving results for the weak Hölder regular solutions, is a commutator estimate which yields a degree formula for the Hölder continuous mapping ∇v , where f , even though being only the very weak Hessian determinant of v , plays the role of the Jacobian $\det \nabla(\nabla v)$. Such commutator estimates were first used in [5] for the Euler equations and in [4] for the isometric immersion problem; this relationship is not surprising in view of the presence of a quadratic term in the equations in all three cases. The same quadratic terms play also a major role for the applicability of the convex integration and iteration techniques in establishing flexibility for the very weak solutions, discussed in the following section.

5. CONVEX INTEGRATION FOR THE MONGE-AMPÈRE EQUATION

The rigidity statements of the previous section should be contrasted with the results explained below. To put such results in a broader perspective, let us recall that questions of rigidity and flexibility of solutions arise in various classes of PDEs in geometry and continuum physics. As a major example, the rigidity of isometric immersions has been studied in differential geometry since already the end of 19th century. It was then known that smooth surfaces in three-dimensional space which are isometric to a piece of plane are developable, i.e. they are locally foliated as a ruled

surface by straight segments aligned at each point with one of the principal directions. Similarly, Hilbert showed that any smooth isometric immersion of the two dimensional sphere into \mathbb{R}^3 must be a rigid motion. The celebrated results of Nash and Kuiper of the mid-20th century highlighted the very fact that these rigidity statements rely much on the regularity of the surface or on the co-dimension of the embedding [24, 14]. They showed that any given Riemannian manifold admits a \mathcal{C}^1 -regular isometric immersion into any Riemannian manifold of one dimension higher, ruling out the above rigid scenarios.

It is perhaps surprising that similar statements on existence of anomalous solutions to the Monge-Ampère equation (3.11) have been missing in the literature. The reformulation (4.1) of the very weak solutions in the context of the nonlinear elasticity of plates and the quadratic structure of the equation (3.13) leads to the following counter intuitive result [22]. Fixing an exponent $\alpha < \frac{1}{7}$, the set of $\mathcal{C}^{1,\alpha}(\bar{\omega})$ solutions to (4.1) is dense in the space $\mathcal{C}^0(\bar{\omega})$, for all the right hand sides $f \in L^{7/6}(\omega)$ defined on an open, bounded, simply connected $\omega \subset \mathbb{R}^2$. More precisely, for every $v_0 \in \mathcal{C}^0(\bar{\omega})$ there exists a sequence $v_n \in \mathcal{C}^{1,\alpha}(\bar{\omega})$, converging uniformly to v_0 and for whom there exists a sequence $w_n \in \mathcal{C}^{1,\alpha}(\bar{\omega}, \mathbb{R}^2)$ converging uniformly to 0 and satisfying (3.13):

$$\frac{1}{2} \nabla v_n \otimes \nabla v_n + \text{sym} \nabla w_n = -\Delta^{-1}(f) \text{Id}_2 \quad \text{in } \bar{\omega}.$$

When $f \in L^p(\omega)$ and $p \in (1, \frac{7}{6})$, the same result is true for any $\alpha < 1 - \frac{1}{p}$.

Similarly to the techniques by Nash and Kuiper, this result is better understood from the convex integration viewpoint, and it is, at the same time, related to the recent applications in the context of fluid dynamics (see [7] and the references therein). Existence of continuous periodic solutions to the three dimensional incompressible Euler equations has been proved in [6]; these solutions dissipate the total kinetic energy. On the other hand, as shown in [5], $\mathcal{C}^{0,\alpha}$ solutions are energy conservative if $\alpha > \frac{1}{3}$. Still, the Onsager's conjecture puts the Hölder regularity threshold for the energy conservation of the weak solutions to the Euler equations at exactly $\mathcal{C}^{0,1/3}$.

The critical value of the Hölder exponent at the threshold of rigidity and flexibility for the $\mathcal{C}^{1,\alpha}$ solutions to the Monge-Ampère equation is not yet clear. This value has been conjectured to be $\frac{1}{3}$, $\frac{1}{2}$ or $\frac{2}{3}$, relying on various intuitive evidences. The Nash-Kuiper iteration method cannot yield anomalous solutions with regularity better than $\mathcal{C}^{1,1/3}$, but on the other hand there seem to be little indication of how to prove the rigidity for the regimes $\frac{1}{3} < \alpha \leq \frac{2}{3}$. Similarly, identifying the characteristics of the $W^{2,p}$ Sobolev solutions of (3.11) for all values of p remains an open problem that could pave the way for better understanding of the behavior of prestrained sheets.

6. THE ENERGY-DRIVEN INVERSE DESIGN

One can use prestrain as a method for designing actively deforming soft matter devices with specific geometries. For example, gel lithography is suggested in [12] as a tool for printing two dimensional sheets which are activated into pre-strained surfaces. The idea is to depart from a minimizer of the dimensionally reduced energy \mathcal{I}_β (as explained in section 3), and inversely calculate the prestrain metric through the Euler-Lagrange equations.

In [29] several sets of experiments are reported, in which nontrivial shapes (see Figure 6.1) were assumed by the actuated sheets of elastomers through writing topological defects, corresponding to singular prestrains, or the nonuniform director profiles through the thickness. The programmable mechanical response of these materials yields monolithic multifunctional devices and serves as reconfigurable substrates for flexible devices.

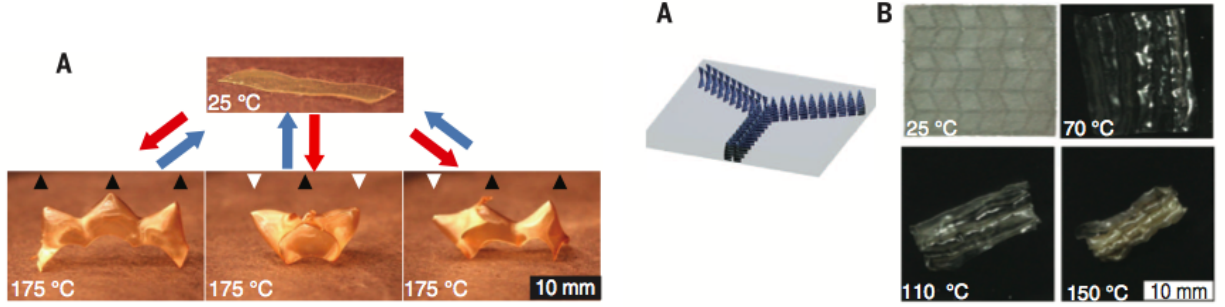


FIGURE 6.1. Actuation by topological defects and the origami-like actuators [29]. Reprinted with permission from AAAS.

To formulate the design problem in a broad sense, suppose one needs to manufacture a two dimensional shell $S \subset \mathbb{R}^3$ such that at each point $p \in S$ a material of the appropriate type p is used. The question is how to print a thin film $\omega \subset \mathbb{R}^2$ combined of these materials, in a manner that the activation of the prestrain in the film would result in a deformation leading eventually to the desired surface shape S . In view of [20, 2], the activation $y : \omega \rightarrow \mathbb{R}^3$ must be an isometric immersion of the Riemannian manifold $(\omega, G_{2 \times 2})$ into \mathbb{R}^3 , where $G_{2 \times 2}$ is the prestrain in the flat (referential) thin film. Denoting the prestrain induced by the material of type p by $g(p)$, this design problem thus requires that $S = y(\omega)$, and that any $x' \in \omega$ carrying a material of type p is mapped to a point $p \in S$:

$$(6.1) \quad y(x') = p, \quad \nabla y(x')^T \nabla y(x') = G_{2 \times 2}(x') = g(p) \quad \text{in } \omega.$$

The same problem can be posed for a three dimensional shape and hence a higher dimensional version of (6.1) is also of interest for potential applications. Indeed, (6.1) leads to a generalization of the isometric immersion problem so far not discussed in the literature.

7. CONCLUSION

In this article, we motivated and showed how the prestrain metric problem can be formulated for three dimensional elastic bodies and how it leads to problems in geometry and analysis of PDEs. In particular, rigidity properties of the weak solutions to geometric PDEs come to the frontline of research, including the discovery of the anomalous solutions to the Monge-Ampère equation. The investigation of the dimensionally reduced models can also shed light on the precise role which is played by the curvature tensor of the prestrain metric in the stress distribution within a three dimensional body and eventually lead to a better understanding of the shape formation phenomena through growth, plasticity, etc. Other largely unexplored related topics include homogenization, symmetry and symmetry breaking, inverse prestrain analysis (useful e.g. in tumour detection) and randomly generated prestrain. These avenues of investigation connect between theory of elasticity, differential geometry, analysis and PDEs. Finally, we hope that a thorough theoretical understanding of the phenomena discussed in this article could help in engineering sheets or bodies with finely controlled shapes, dynamics, structural resistance to loads and elastic properties such as rigidity and flexibility.

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