A report on topological singularities of maps in function spaces between manifolds

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Our intention in this thesis is to study some important problems related to the "topological singular set" of the maps in a given function space. This object, which is to be defined without ambiguity for some categories of these mappings, is the obstruction which characterizes the non-approximability of a mapping in this space by the smooth mappings. These topological singularities and their properties are the base of some interesting results on the weakly harmonic maps into the sphere or on the weak or strong density of smooth maps in function spaces. They have become an independent subject of study with important questions to solve, related to different domains such as Functional Analysis, Geometric Measure Theory, Topology and Geometry.

Here we limit ourselves to the Sobolev spaces between manifolds. But remark that the same problems are worthy to ask for any other function space. Consider two compact riemannien manifolds M and N of respective dimensions n and k, such that N is closed and isometrically embedded in some euclidien space $\mathbb{R}^N$. for $p \geq 1$, the Sobolev space $W^{1,p}(M, N)$ is defined by

$$W^{1,p}(M, N) := \{u \in W^{1,p}(M, \mathbb{R}^N);\, u(x) \in N \text{ p.p. dans } M\}.$$

This space is equipped with the induced weak and strong topologies of $W^{1,p}(M, \mathbb{R}^n)$ and is closed under the convergence in these topologies. The p-energy functional is defined by $E_p(u) := \int_M |\nabla u|^p$ and is called the Dirichlet energy $E(u) := \int_M |\nabla u|^2$ for $p = 2$. Also, for a map $\varphi \in C^\infty(\partial M, N)$ we set

$$W^{1,p}_\varphi(M, N) := \{u \in W^{1,p}(M, N);\; u|_{\partial M} = \varphi\}.$$

For definitions concerning the Geometric Measure Theory the reader can refer to [16] or [28]. Meanwhile, we will refer to integer multiplicity rectifiable currents (respectively real multiplicity currents) with finite mass by the term i.m. rectifiable (respectively normal) currents.

Harmonic mappings into the sphere

Let us begin with a variational problem which leads us, in a natural way, to problems related to topological singularities.

Consider the Sobolev space $H^1(\Omega, S^2)$ where $\Omega \subset \mathbb{R}^n$, $n \geq 3$, is a bounded open set and S^2 is the 2-dimensional unit sphere in $\mathbb{R}^3$. We call u a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt} E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C_c^\infty(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions :

$$\begin{cases} -\Delta u = u|\nabla u|^2 \quad \text{in} \quad \Omega \\ \\ u(x) \in S^2 \quad \text{a.e.} \end{cases}$$

Let $\varphi : \partial\Omega \to S^2$ be a smooth map which has a regular extension into Ω. The existence of a weakly harmonic map equal to φ on the boundary can be easily proved by a straightforward minimizing argument. By the way, the uniqueness and regularity questions for weakly harmonic maps in $H^1_\varphi(\Omega, S^2)$ have not the same answers as in the classic cases, i.e. when the target manifold is an euclidean space.

The smoothness of harmonic extensions into S^2

One of the important problems which is still open is if smooth harmonic extensions of φ into Ω exist. In the first step one may want to minimize the Dirichlet energy in $H^1_\varphi(\Omega, S^2)$ and prove the regularity of the solution. But in fact if we define

$$\mu_\varphi := \inf_{H^1_\varphi(\Omega,S^2)} E(u) \leq \inf_{C^\infty_\varphi(\bar\Omega,S^2)} E(u) =: \bar\mu_\varphi\,,$$

the strict inequality

$$\mu_\varphi < \bar\mu_\varphi$$

happens sometimes (See [22]). Thus minimizers of E are not necessarily smooth and we should find other harmonic maps which could be a suitable candidate for a smooth solution. Meanwhile R.Schoen and K.Uhlenbeck ([35]) proved that these minimizers are smooth in Ω except on a finite set of points.

In trying to attack this problem, another functional on $H^1_\varphi(\Omega, S^2)$ has been studied which is called the "relaxed energy". In fact, the relaxed energy is the largest sequentially lower semi-continuous functional on $H^1_\varphi(\Omega, S^2)$ which is less than E on $C^\infty_\varphi(\Omega, S^2)$:

Définition 1 *The relaxed energy $\mathcal{F}$ of E on $H^1_\varphi(\Omega, S^2)$ is defined to be*

$$\mathcal{F}(u) := \inf\left\{ \liminf_{n\to\infty} E(u_n)\,;\; u_n \in C^\infty_\varphi(\Omega, S^2)\,,\; u_n \rightharpoonup u \right\}. \tag{0.1}$$

Since the smooth maps which take φ as their boundary value are weakly sequentially dense in $H^1_\varphi(\Omega, S^2)$ (See [2]), we observe that $\mathcal{F}$ is well defined. Moreover $\mathcal{F}$ is sequentially lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and we have

$$\inf_{H^1_\varphi(\Omega,S^2)} \mathcal{F} = \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{0.2}$$

This equation shows the importance of study of $\mathcal{F}$. Since the infimum of $\mathcal{F}$ in $H^1_\varphi(\Omega, S^2)$ is achieved, the question which should be considered then is whether a minimizer of $\mathcal{F}$ is weakly harmonic and to what extent it is regular.

In this line, F.Bethuel, H.Brezis, J.M. Coron and E.Lieb (See [5] and [10]) showed the striking fact that, for $n = 3$, the relaxed energy achieves the following elegant algebric formula :

$$\mathcal{F}(u) = F(u) := E(u) + 8\pi L(u) \tag{0.3}$$

where

$$L(u) := \frac{1}{4\pi} \sup_{\substack{\psi\,:\,\Omega\to\mathbb{R} \\ |d\psi|_\infty \le 1}} \left\{ \int_\Omega u^*\omega_V \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega_V \wedge \psi \right\} \tag{0.4}$$

where ω_V is the volume form on S^2 (or can be replaced by any 2-form ω, $\int_{S^2} \omega = 4\pi$). In particular this yields that the critirical points of $\mathcal{F}$ are weakly harmonic. F.Bethuel and H.Brezis showed also that the minimizers of $\mathcal{F}$ are smooth in Ω except on a finite set of points (See[4]).

The intuitive approach for $L(u)$ is that if $u \in H^1_\varphi(\Omega, S^2)$ is smooth in Ω except on a set of finite points $\{p_1, ..., p_m\}$, taking the degree d_i on the point p_i, then $L(u)$ is the minimum length of the segments connecting these singularities with respect to the multiplicities (See [10]). In other words

$$L(u) = m_i\left(\sum_{i=1}^m d_i\,[[p_i]],\, \Omega \right)$$

where $m_i(\Omega, S^2)$, for the i.m. rectifiable 0-current $\mathbf{S}$, is defined by

$$m_i(\mathbf{S}, \Omega) := \inf\left\{ \mathbf{M}(\mathbf{T})\,;\, T \in \mathcal{R}_1(\mathbb{R}^3),\ \mathrm{spt}\,\mathbf{T} \subset \overline{\Omega}\,,\, \partial\mathbf{T} = \mathbf{S} \right\}.$$

In the first chapter, we study the same approach for $n > 3$ but this generalisation meets obstacles. One may introduce for ω, any 2-form on S^2 which satisfies $\int_{S^2} \omega = 1$:

$$L(u) := \sup_{\substack{\psi \in \Omega^{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \le 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{0.5}$$

as a generalization of $L(u)$ in the 3-dimensional case. Observe that L is independant of the choice of ω and is continuous on $H^1_\varphi(\Omega, S^2)$ for $\Omega \subset \mathbb{R}^n$ and the functional

$$F(u) := E(u) + 8\pi L(u) \tag{0.6}$$

would still be weakly lower semi-continuous. But we have this theorem :

Theorem 1 *For every $\Omega \subset \mathbb{R}^4$ and every map $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, there exists $u \in H^1_\varphi(\Omega, S^2)$ such that*

$$F(u) < \mathcal{F}(u)\,.$$

Moreover there exists a domain $\Omega \subset \mathbb{R}^4$ and $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable onto Ω, for which this gap phenomenon exists :

$$\inf_{H^1_\varphi(\Omega,S^2)} E < \inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E\,.$$

■

The difference with the case $n = 3$ lies in the value which $L(u)$ represents. We shall consider a map $u \in H^1_\varphi(\Omega, S^2)$, which is smooth except on a finite union of $(n-3)$-dimensional submanifolds of Ω : $\{\sigma_1, ...\sigma_m\}$ (We say that $u \in R^\infty(\Omega, S^2)$). The degree d_i of u on each σ_i is well defined and we set the topological singularity of u, $\mathbf{S}_u$, to be

$$\mathbf{S}_u := \sum_{i=1}^{m} d_i\,[[\sigma_i]]. \tag{0.7}$$

Calculating $L(u)$, we see that

$$L(u) = \sup_{|d\psi|_\infty \le 1} \int_{\mathbf{S}_u} \psi \le \sup_{\|d\psi\|^*_\infty \le 1} \int_{\mathbf{S}_u} \psi = m_r(\mathbf{S}_u, \Omega) \tag{0.8}$$

where $\|.\|^*$ is the co-mass norm on the space of forms and

$$m_r(\mathbf{S}_u, \Omega) := \inf\left\{\mathbf{M}(\mathbf{T})\,;\, \mathbf{T} \in \mathcal{D}_{n-2}(\mathbb{R}^n)\,, \partial\mathbf{T} = \mathbf{S}_u,\, \mathrm{spt}\,\mathbf{T} \subset \overline{\Omega}\right\}$$

is the mass of the minimal normal (real) current in Ω with boundary $\mathbf{S}_u$.

Meanwhile, $m_i(\mathbf{S}_u, \Omega)$, the minimal mass of i.m. rectifiable currents in $\overline{\Omega}$ which are bounded by $\mathbf{S}_u$, is still proportional to the energy needed for removing the singularities of u and estimating it weaky by smooth maps (See the further proposition 1). Here arises the main question which should be answered if we want to continue as above, that is if

$$m_r(\mathbf{S}, \Omega) = m_i(\mathbf{S}, \Omega) \quad \forall \mathbf{S} \in \mathcal{R}_{n-3}(\Omega).$$

But contrary to the case $n = 3$, the answer is no for $n > 3$. Specially, for $n = 4$, there exists a curve $[[\Gamma]]$ in $\mathbb{R}^4$ for which

$$m_r([[\Gamma]]) \le \frac{1}{2} m_i(2[[\Gamma]]) < m_i([[\Gamma]]).$$

This gap phenomenon was firstly proved by L.C.Young in [42]. F.Morgan in [27] and B.White in [37] have given other examples of such curves in $\mathbb{R}^4$.

Remark 1 *We have always the relation*

$$\sup_{\substack{\|d\psi\|_\infty^* \leq 1 \\ spt\psi \subset \overline{\Omega}}} \int_{\mathbf{S}} \psi = m_r(\mathbf{S}, \Omega)$$

which is due to the fact that there exists always a calibration for minimizing normal currents (See Chapter I for the references).

The topological singularities and the relaxed energy

The question which arises then is to find the equivalent formula for (0.3) for the relaxed energy when $n > 3$. Regarding (0.8), we can consider L, for $n = 3$ as a continuous extension of $m_i(\mathbf{S}_u, \Omega)$ $(= m_r(\mathbf{S}_u, \Omega))$ into all $H^1(\Omega, S^2)$. Therefore, for generalizing the result to higher dimensions, one should extend the definition of the topological singularities over $H^1(\Omega, S^2)$:

Définition 2 *Let $u \in H^1_\varphi(\Omega, S^2)$. We define the topological singularity of u to be the current $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ defined by*

$$\mathbf{S}_u(\alpha) := \int_\Omega u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega).$$

Here $\mathcal{D}^k(\Omega)$ is the set of smooth k-forms on Ω with compact support (See[16], 2.2.3) and ω is some 2-form on S^2 for which $\int_{S^2} \omega = 1$.

Remark 2 *F.Béthuel, J.M.Coron, F.Demengel et F.Hélein ([6]) proved that "$\mathbf{S}_u = 0$" is the necessary and sufficient condition for $u \in H^1(\mathbf{B}^n, S^2)$ to be approximable by smooth maps in the strong topology. This is the reason behind the choice of "topological singularity" as the name for $\mathbf{S}_u$.*

This definition coincides with the one given for R^∞ maps in (0.7) (See [16], vol II section 5.4.2. The reader can also find the detailed proof of this fact in Chapter II). Observe that the expression $m_i(\mathbf{S}_u, \Omega)$ has a meaning for any $u \in H^1_\varphi(\Omega, S^2)$ only if S_u is a boundary for an i.m. rectifiable current. Although this necessary condition is satisfied for $n = 3$, the proof for $n > 3$ is not the same and we are forced to use the methods developed in [16] for the cartesian currents to prove it. The difficulty lies on the fact that the question of the strong continuity of m_i for $n > 3$, even over $R^\infty(\Omega, S^2)$, is still open. This is also the obstacle to identify the functional

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega)$$

with the relaxed energy. Is $\mathcal{F} = \widetilde{F}$? ∎

Precisely we have this proposition proved in Chapter II :

Proposition 1 *Let $u \in H^1_\varphi(\Omega, S^2)$, then $\mathbf{S}_u$ is the boundary of some i.m.rectifiable current. Set*

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega).$$

$\widetilde{F}$ is lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and its critical points are weakly harmonic. Moreover

$$\widetilde{F}(u) \leq \mathcal{F}(u)\,, \quad \forall u \in H^1_\varphi(\Omega, S^2).$$

■

We will talk about the problem of topological singularities for maps into spheres in a more general context.

The multiplicity of S^2-valued harmonic extenstions

In the second chapter, we will answer to the question of multiplicity of harmonic extensions into S^2 for a smooth mapping $\varphi : \Omega \to S^2$, $n = \dim\Omega > 3$. Here is the theorem we prove in this chapter.

Theorem 2 *Let Ω be a regular bounded domain in $\mathbb{R}^n$, $n > 3$, and φ a non-constant smooth map from $\partial\Omega$ into S^2. Then φ admits infinitely many weakly harmonic extensions.*

■

Remark 3 *This result is independent of the choice of the metric on S^2 .*

In [21], R. Hardt, D.Kinderlehrer and F.H.Lin had proved the existence of infinitely many weakly harmonic extensions to an axially symmetric boundary condition in $H^1(\mathbf{B}^3, S^2)$ where $\mathbf{B}^3$ is the unit ball in $\mathbb{R}^3$. The method consists in constructing a non-axially symmetric harmonic extension and then one obtains infinitely many different harmonic maps with the same boundary data by rotating this extension around the symmetry axis.

Another method consists in finding new weakly harmonic maps minimizing the variants of the relaxed energy already presented in this Introduction. This has been done by F.Bethuel, H.Brezis and J.-M.Coron in [5] where they introduced such functionals which they called "relaxed energies". Using these functionals they proved for $n = 3$ that if φ is not homotopic to a constant or if we have this gap condition

$$\min_{H^1_\varphi(\Omega,S^2)} E(u) < \inf_{C^\infty_\varphi(\Omega,S^2)} E(u)$$

then φ admits infinitely many weakly harmonic extensions inside Ω. Using the same gap condition, T.Isobe proved the corresponding result for the case $n \geq 4$ in [26], still using the relaxed energies whose definition was extended to higher dimensions.

At last, using his *strict dipole insertion lemma*, (the 3-dimensional version of the furthur lemma 1) proved in [32], T.Rivière showed that if Ω is a regular bounded domain of $\mathbb{R}^3$, a non constant smooth boundary data $\varphi : \partial\Omega \to S^2$ admits always infinitely many weakly harmonic extensions (Appeared in [33]).

Let us consider the same method for $n \geq 4$. Although the F-energy presented in (0.6) is not the relaxed energy, its minimizers are still weakly harmonic. Proving this fact in Chapter II, we will produce new weakly harmonic maps using this energy. But the difficult step consists in finding some equivalent construction in any dimensions of the insertion of 2 singular points with the strict inequality like in [32] for $n = 3$. In the first sight it seems that we should insert this time a couple of singularities of dimesnion $n - 3$ (e.g. two circle-singularities for when $n = 4$). But the dipole for $n = 3$ is nothing else than the sphere $S^0 = S^{n-3}$ in 3 dimensions. So it appears that ([32], lemma A.1) can be generalized by inserting this time an $(n-3)$-dimensional singular sphere. This lemma, technically more involved than the 3 dimensional case, is the main step to prove theorem 2.

Lemme 1 *Let Ω be a bounded regular domain in $\mathbb{R}^n$ and u a regular non-constant map from Ω to S^2. Let x_0 be a point of Ω for which $\nabla u(x_0) \neq 0$. Then for every $\rho > 0$ there exists a map $v \in H^1(\Omega, S^2)$ and $0 < \delta < \rho$ such that*

(i) $v = u$ *on* $\Omega \backslash B_\rho(x_0)$

(ii) $\mathbf{S}_v = [[\sigma]]$

(iii) $E(v) < E(u) + 8\pi\omega_{n-2}\delta^{n-2} = E(u) + 8\pi L(v, u)$

where σ is an $(n-3)$-dimensional sphere of center x_0 and radius δ and ω_k is the volume of the unit k-dimensional disk.

■

Topological singularities in $W^{1,p}(M, S^p)$

Considering the characteristics of $\mathbf{S}_u$, the topological singular set defined in the previous section for any map $u \in H^1_\varphi(\Omega, S^2)$, it is interesting and natural to consider the problem of the topological singular set $\mathbf{S}_u$ for the maps $u \in W^{1,p}(M, S^p)$ when p is any integer and M a compact manifold.

Any map $u \in W^{1,p}(\mathbf{B}^n, S^p)$, $p < n$, is the strong limit of smooth maps if and only if $d(u^*\omega_{S^p}) = 0$ in the sense of distributions (See [6]). By this, we can generalize definition 2 for this space. :

Définition 3 *Let $u \in W^{1,p}(M, S^p)$. We define the "local" topological singularity of u, $\mathbf{S}_u \in \mathcal{D}_{n-p-1}(M)$, to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_M u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-p-1}(M).$$

Here $\mathcal{D}^k(M)$ is the set of smooth k-forms on M with compact support (*See*[*16*], 2.2.3) *and ω is some p-form on S^p for which $\int_{S^p} \omega = 1$.*

We recall that $m_i(\mathbf{S})$ (resp. $m_r(\mathbf{S})$) is the minimal mass of i.m. rectifiable (resp. normal) currents supported in M and bounded by $\mathbf{S}$. Two questions regarding the topological singularities in $W^{1,p}(M, S^p)$ are still open for almost all values for p : Is $\mathbf{S}_u$ the boundary of some i.m. rectifiable current, when M is a closed manifold? ■

Assume that the answer to the previous question is positive. Then, do $m_i(\mathbf{S}_{u_m})$ tend to $m_i(\mathbf{S}_u)$ if $u_m \to u$ strongly in $W^{1,p}$? ■

Minimal normal and i.m. rectifiable currents

$\mathbf{S}_u$ is effectively the boundary of some normal current in $W^{1,p}(M, S^p)$ and

$$m_r(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$$

for any convergent sequence in $W^{1,p}(M, S^p)$. As a consequence, if for any i.m.rectifiable current $\mathbf{S}$ of dimension $n - p - 1$, $m_i(\mathbf{S}) \leq C m_r(\mathbf{S})$ for some constant $C > 0$, the two above questions will have positive answers. But, except for $p = 1$ or $p = n - 1$, we do not know if this constant exist. Assume that $k \neq 0, n - 2$. Is the quantity

$$\frac{m_i(\mathbf{S})}{m_r(\mathbf{S})} = \sup_{l \in \mathbb{N}} \frac{l m_i(\mathbf{S})}{m_i(l\mathbf{S})},$$

equi-bounded uniformly over all i.m. rectifiable k-currents supported in a compact subset of $\mathbb{R}^n$? ■

Remark 4 *As we already mentioned, $m_i(\mathbf{S}) = m_r(\mathbf{S})$ is always true if $k = 0, n - 2$ (For the references see the discussion about the calibrations in the first chapter).*

A geometric interpretation for $\mathbf{S}_u$

M.Giaquinta, G.Modica and J.Soucek gave another definition for $\mathbf{S}_u$, which is equivalent to the ours (See [16], vol II, section 5.4.2). $\mathbf{S}_u$ is defined to be the horizontal part of ∂G_u, when G_u is the rectifiable graph of u, considered as a cartesian current in $M \times S^p$ (See [16], vol I). Considering this fact and using the characteristics of the cartesian currents and the polyconvex envelopes of the Dirichlet energy discussed in [16], we proved that the Question has a positive answer for $p = 2$ (See the above proposition 1).

And if S^p is an H-space?

In Chapter III we answer to the two above questions regarding the topological singularities of $u \in W^{1,p}(M, S^P)$ for $p = 3$ and 7. The particularity of these two cases reside in the fact that S^3 and S^7 (alongside with S^1 and S^0) are the only spheres which are H-spaces, i.e. there is a smooth multiplication

$$\kappa : S^p \times S^p \to S^p$$

such that the induced homotopic homeomorphism

$$\kappa_* : \pi_p(S^p) \oplus \pi_p(S^p) \to \pi_p(S^p)$$

is the sum of elements in $\pi_p(S^p)$ ([8], section VI.15). As a result, the method we use does not work for other values of p. Here is our main result

Theorem 3 *Let $p = 3$ or 7, $p < n = dim\, M$ and $u \in W^{1,p}(M, S^p)$, $\partial M = \emptyset$. Then $\mathbf{S}_u$ is the boundary of an i.m. rectifiable current in M. Moreover, $m_i(\mathbf{S}_{u_m} - \mathbf{S}_u) \to 0$ if the u_m converge strongly to u in $W^{1,p}(M, S^p)$.*

■

A new perspective for the topological singularities

In the last chapters of this thesis we will try to generalize the notion of the topological singular set for certain categories of Sobolev spaces $W^{1,p}(\mathbf{B}^n, N)$. We will explain how these efforts let us to prove some theorems about the sequentially weak density of smooth maps in these spaces. We will use locally lipschitz projections of N over its $[p]$-skeletons, the results of F.J.Almgren, W.Browder and .Lieb about the inverse images for the Sobolev maps into spheres ([1]) and the singularity removing propositions adapted to our situation. We recall that the topological singularities should be defined to identify the obstruction to the non-approximability of a Sobolev map between M and N by the smooth maps from M into N. The singularities we consider detect the local obstructions of the approximability, therefore we will emphasize in the Chapter IV on the case $M = \mathbf{B}^n$, where $\mathbf{B}^n$ is the n-dimensional unit disk. F.Hang and F.H.Lin [20] have recently showed the possible existence of "global" obstructions when the topology of the domain M is not trivial. So one should be careful when considering the Sobolev spaces $W^{1,p}(M, N)$ for generic compact smooth manifold M.

Flat chains with coefficients in normed groups

In [7], F.Bethuel and X.Zheng proved that smooth maps are not dense in $W^{1,p}(\mathbf{B}^n, N)$, if $p < n$ and $\pi_{[p]}(N) \neq 0$. In this case, the best one can do is to approximate the maps in $W^{1,p}(\mathbf{B}^n, N)$ by maps which are smooth away from a finite union $\Sigma = \bigcup_{i=1}^r \Sigma_i$ of smooth

$(n-p-1)$-dimensional submanifolds of $\mathbf{B}^n$. This set of maps is called $R^{\infty,p}(\mathbf{B}^n, N)$. A map $v \in R^{\infty,p}(\mathbf{B}^n, N)$ realizes elements σ_x of $\pi_{[p]}(N)$ on the $[p]$-spheres centered at any point $x \in \Sigma(v)$ and contained in the normal $[p]+1$ plane to $T_x\Sigma(v)$. If for some $x \in \Sigma(v)$, σ_x is non trivial, then v can not be approximated by smooth maps in the strong topology (See [2]).

As an example, the smooth maps are not dense in $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ since $\pi_1(\mathbb{RP}^2) \neq 0$. Then, $v \in R^{\infty,1}(\mathbf{B}^2, \mathbb{RP}^2)$ is smooth except on a finite number of points in $\mathbf{B}^2 : \{p_1, \ldots, p_r\}$. If v has the non-zero homotopy type of $\pi_1(\mathbb{RP}^2) = \mathbb{Z}_2$ around one of these points, it can not be approximated by the smooth maps in $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ (We can construct such v). The idea is then to identify and define properly the "topological singular set" of such v, which allows us to extend the definition to any map $u \in W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$.

The usual method, using the differential forms and proposed by F.Bethuel, J.M.Coron, F.Demengel and F.Hélein in [6] is not helpful since $\pi_1(\mathbb{RP}^2)$ is not torsion free and the homotopy cycles in $\mathbb{RP}^2$ are not detected by the 1-forms. For the same reasons, the approach of ([16], vol II, section 4.4.2) by M.Giaquinta, G.Modica and J.Soucek, using the graph of Sobolev maps is not satisfactory.

The idea would be to use the flat chains with coefficients in a normed abelian group G, which are the generalizations of normal ($G = \mathbb{R}$) and rectifiable ($G = \mathbb{Z}$) currents. This theory was first introduced by H.Federer [13] and W.Fleming [15]. Recently there have been remarkable advances by B.White ([38] and [39]). In fact, we can imagine the topological singular set of v, $\mathbf{S}_v$, as a 0-chain with coefficients in $\mathbb{Z}_2$:

$$\mathbf{S}_v := \sum_{i=1}^{r} \sigma_{p_i}[[p_i]] \quad (\sigma_{p_i} := [v(\partial B_\delta(p_i))]_{\pi_1(\mathbb{RP}^2)} \in \mathbb{Z}_2).$$

The question would be to understand the behaviour of $\mathbf{S}_{v_m}$ for a convergent sequence $v_m \to u \in W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$ and possibly to prove a convergence of the chains $\mathbf{S}_{v_m}$ in the flat norm to some $\mathbb{Z}_2$-chain we would call the topological singularity of u.

Naturally, regarding what we mentioned about the realizations of elements of $\pi_{[p]}(N)$ by $v \in R^{\infty,p}(\mathbf{B}^n, N)$ around its singularities, we can proceed in the same way for maps in $W^{1,p}(\mathbf{B}^n, N)$, i.e. to define the topological singularities of $v \in R^{\infty,p}(\mathbf{B}^n, N)$ as a $\pi_{[p]}(N)$ -chain and to study the behaviour of these chains for the convergent sequences $v_m \to u \in W^{1,p}(\mathbf{B}^n, N)$. Nevertheless, this program is not suitable for all Sobolev spaces, as shows the example $W^{1,3}(\mathbf{B}^4, S^2)$ treated by R.Hardt et T.Rivière (See [24]).

In Chapter IV, we will prove this theorem :

Theorem 4 *Assume that $\mathbf{B}^n$ is the unit disk in $\mathbb{R}^n$ and that N is a compact riemannian manifold of dimension $k \geq [p]$, $\partial N = \emptyset$. We assume also that either $[p] = 1$ and $\pi_1(N)$ is* *abelian, or $[p] = 3, 7, n-1$ and N is $([p]-1)$-connected, i.e.*

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0.$$

Then $\mathbf{S}_u$, the topological singularity of $u \in W^{1,p}(\mathbf{B}^n, N)$ is well defined as a flat $\pi_{[p]}(N)$-chain and the flat norm of $\mathbf{S}_{u_m} - \mathbf{S}_u$ converge to zero if $u_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$. Moreover, u is the strong limit of maps in $C^\infty(\mathbf{B}^n, N)$ if and only if $\mathbf{S}_u = 0$. Also, if $u|_{\partial \mathbf{B}^n} = \varphi$ is smooth and smoothly extendable over $\mathbf{B}^n$, $\mathbf{S}_u$ will be the boundary of some flat $\pi_{[p]}(N)$-chain of finite mass (and as a result of rectifiable support) and "$\mathbf{S}_u = 0$" would be the necessary and sufficient condition for u to be the strong limit of maps in $C^\infty_\varphi(\mathbf{B}^n, N)$.

■

Remark 5 *Regarding $W^{1,1}(\mathbf{B}^2, \mathbb{RP}^2)$, we have this remarkable fact that we can identify $\mathbf{S}_u$ for any map u in this space to a $\mathbb{Z}_2$-valued Borel measure of total finite variation. The reader can refer to [38] where B. White give the conditions on G for which a finite mass flat G-chain has a rectifiable support.*

We should add some other remarks. First, the reason we can not state the same results for all values of $[p]$ is what we explained in the previous section, i.e. $[p] = 1, 3, 7$ and $n-1$ are the only values for which there is a proof for the integral flat convergence of the topological singularities of a convergent sequence in $W^{1,[p]}(\mathbf{B}^n, S^{[p]})$. Second, we can extend these results for $[p] = 3, 7$ and $n-1$, even if $\pi_1(N) \neq 0$, under certain conditions (See the proof in Chapter IV). At last, we should recall that there are examples of $([p]-1)$-connected manifolds whose $[p]$-th homotopy group is not torsion free otherwise the cases we consider would reduce to those already studied in [6]. As an exmaple, the Stiefel manifolds $V_k(\mathbb{R}^n)$, when $n-k$ is odd, are $(n-k-1)$-connected and $\pi_{n-k}(V_k(\mathbb{R}^n)) = \mathbb{Z}_2$ is not torsion free (See [25])

F.Hang and F.H.Lin [20] have found examples where the absence of the local obstructions in not sufficient for that a map $u \in W^{1,p}(M, N)$ be strongly approximable by smooth maps. Precisely, there is a map in $H^1(\mathbb{CP}^2, S^2)$ for which $d(u^*\omega) = 0$ but u is not in the strong closure of smooth maps in this space. Also there are maps in $W^{1,3}(\mathbb{CP}^3, \mathbb{CP}^2)$ which are not strongly approximable by smooth maps though $\pi_3(\mathbb{CP}^2) = 0$. The necessary and sufficient conditions for that a Sobolev map between two manifolds be approximable by the smooth maps are still unknown for the general case.

Finally we ask this question for which we have no definite answer : How should one define the topological singular set of maps in $W^{1,1}(\mathbf{B}^n, N)$ when $\pi_1(N)$ is not abelian? The same question can also be asked about the functional spaces $H^{\frac{1}{2}}(M, N)$. ■

In Chapter V, when we consider the problem of weak density of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$ for non-abelain fundamental group, we will try to explain the obstacles regarding this situation and to put the bases for a future response to this question.

Weak density of smooth maps and the connections

While the question of flat convergence for the singularity chains of a sequence of convergent maps $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ remain to be answered (See theorem 4), one can ask also a weaker question : Does the flat norm of $\mathbf{S}_{v_m}$ remain bounded as $v_m \to u$? This is another problem we address in Chapter IV about the uniform boundedness of the mass $\mathbf{M}(\mathbf{T}_m)$ of a minimal connection $\mathbf{T}_m$ ($\partial \mathbf{T}_m = \mathbf{S}_m$) as $v_m \to u$.

Related to this question is the problem of the weak density of smooth maps in $W^{1,p}(M, N)$. Although the density of smooth maps for the weak topology can be easily handled from the one for the strong topology (See [2]), the question of the density of smooth maps in $W^{1,p}(M, N)$ for the sequentially weak topology , where $p \in \mathbb{N}$, is more involved : For $p \in \mathbb{N}$, $\pi_p(N) \neq 0$, does there exist for any $u \in W^{1,p}(M, N)$ a sequence $u_m \in C^\infty(M, N)$ such that $u_m \rightharpoonup u$ in $W^{1,p}$? The case $M = \mathbf{B}^3$, $N = S^2$, $p = 2$ was treated by F.Bethuel, H.Brezis, J.M.Coron and E.Lieb in [10], and [3]. F.Bethuel mentioned that the answer is yes for $M = \mathbf{B}^n$, $N = S^p$, $p \geq 2$ in [2]. In [19], P.Hajlasz proved that if N is $(p-1)$-connected, any map in $W^{1,p}(M, N)$ is the weak limit of a sequence of smooth maps in this space. Observe that this result can be also deduced from the work of F.Bethuel, J.-M.Coron, F.Demengel and F.Hélein in [6] for when $M = \mathbf{B}^n$ and $\pi_p(N)$ is torsionless.

As we will explain in Chapters IV and V, the control of the mass of the minimal chain connecting $\mathbf{S}_{v_m}$ for $v_m \in R^{\infty,p}(\mathbf{B}^n, N)$ converging strongly to u permits to give a positive answer to the sequentially weak density of smooth maps. This appraoch is different from the one used by P.Hajlasz and can be used for proving his theorem and some other partial results regarding the weak sequential denstiy of maps in $W^{1,p}(\mathbf{B}^n, N)$. Specially, Hajlasz's method is not adapted when we wish to approach $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$ in the weak topology by a sequence of maps in $C^\infty_\varphi(\mathbf{B}^n, N)$ (He does not mention this question in [19]). The case $p = 1$ is more involved when $\pi_1(N)$ is non-abelian and we will discuss it in an independent chapter (Chapter V). The reason is that in this case we can not identify an element of $\pi_1(N)$ without fixing its base point, so defining the topological singularities as the flat chains with coefficeints in $\pi_1(N)$ meets obstacles. There are some other technical complications which we will mention in Chapter V.

In Chapter IV and in Chapter V we will prove :

Theorem 5 *Assume that $\mathbf{B}^n$ is the unit disk in $\mathbb{R}^n$ and that N is a compact riemannien manifold of dimension $k \geq [p]$, $\partial N = \emptyset$, and either $[p] = 1$ or N is $([p]-1)$-connected, i.e.*

$$\pi_1(N) = \cdots = \pi_{[p]-1}(N) = 0.$$

Also assume that $\varphi : \partial \mathbf{B}^n \to N$ admits a smooth extension over $\mathbf{B}^n$. Then, for any map $u \in W^{1,p}_\varphi(\mathbf{B}^n, N)$, there exists a sequence of smooth maps $u_m : \mathbf{B}^n \to N$, $u_m|_{\partial \mathbf{B}^n} = \varphi$, such that $\|u_m - u\|_{L^p} \to 0$ and that $\|u_m\|_{W^{1,p}}$ is bounded by a constant.

■

Remark 6 *Naturally if $p \geq 2$, we can always find a subsequence of such a sequence, converging weakly to u. But the question of sequentially weak density of smooth maps in $W^{1,1}(\mathbf{B}^n, N)$ is still open.*

We can extend the results of theorem 5 for $p \geq 2$ when $\pi_2(N)$ is finitely generated. Specially for $p = 2$

Theorem 6 *If $\pi_2(N)$ is finitely generated, we have the sequentially weak density of $C^\infty(\mathbf{B}^n, N)$ (resp. $C^\infty_\varphi(\mathbf{B}^n, N)$) in $H^1(\mathbf{B}^n, N)$ (resp. $H^1_\varphi(\mathbf{B}^n, N)$).*

■

The recent developments by F.Hang and F.H.Lin [20] have shown that one should consider the global topology of M for extending these results to any smooth compact manifold M as the domain using the same methods. We hope to expose in near future how our proofs for the sequentially weak density of smooth maps in the Sobolev spaces can be adapted to any domain.

Remark 7 *We do not have always the equi-boundedness of the mass of minimal connections for $\mathbf{S}_{v_m}$ when $v_m \to u$ in $W^{1,p}(\mathbf{B}^n, N)$: For instance, there exist $v_m \in R^{3,\infty}(\mathbf{B}^4, S^2)$ such that*

$$\inf \{\mathbf{M}(\mathbf{T}_m)\, ;\ \mathbf{T}_m \textit{ is a } \mathbb{Z} - \textit{chain such that } \partial \mathbf{T}_m = \mathbf{S}_{v_m}\} \longrightarrow +\infty$$

as $v_m \to u$ in $W^{1,3}(\mathbf{B}^4, S^2)$ (See [24]). However it is not excluded that the smooth maps be sequentially weakly dense in $W^{1,3}(\mathbf{B}^4, S^2)$.

Are the smooth maps sequentially weakly dense in $W^{1,3}(\mathbf{B}^4, S^2)$? Also, regarding the results obout the sequentially weak density of smooth maps in $H^{\frac{1}{2}}(S^2, S^1)$ by T.Rivière ([34]), we ask the same questions about $H^{\frac{1}{2}}(\mathbf{B}^n, N)$. ■

Remark 8 *Meanwhile, using a global obstruction, F.Hang and F.H.Lin proved that the smooth maps are not sequentially dense in $W^{1,3}(\mathbb{CP}^2, S^2)$ (See[20]) .*

This question remains open too for some other cases which are not put forward in this thesis.

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