Relaxing the Dirichlet energy for maps into S^2 in high dimensions

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Abstract

We prove the existence of a gap phenomenon, non-existent for the 3 dimensional case, for the relaxed Dirichlet energy of maps from a 4-dimensional domain into sphere.

1 Introduction

Let $\Omega \subset \mathbb{R}^n$ be a bounded open set with regular boundary and let

$$H^1(\Omega, S^2) = \{u \in H^1(\Omega, \mathbb{R}^3)\,;\, u(x) \in S^2 \quad \text{a.e. on}\,\Omega\}$$

and

$$H^1_\varphi(\Omega, S^2) = \{u \in H^1(\Omega, S^2)\,;\, u = \varphi \quad \text{on}\,\partial\Omega\}$$

where φ is a given boundary data. For $u \in H^1_\varphi(\Omega, S^2)$ the Dirichlet energy is given by $E(u) = \int_\Omega |\nabla u|^2$. We assume that φ is in $C^\infty(\partial\Omega, S^2)$ and can be extended into Ω by a smooth map.

We say that u is a weakly harmonic map if it is a critical point for the functional E, i.e. if and only if we have

$$\frac{d}{dt} E\left(\frac{u+tv}{|u+tv|}\right)_{|t=0} = 0 \quad \text{for all} \quad v \in C^\infty_c(\Omega, \mathbb{R}^3)\,.$$

In other words, u is weakly harmonic in the Sobolev space $H^1(\Omega, S^2)$ if it satisfies the following equation in the sense of distributions :

$$\begin{cases} -\Delta u = u|\nabla u|^2 \quad \text{in} \quad \Omega \\ u(x) \in S^2 \quad \text{a.e.} \end{cases} \tag{1.1}$$

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Assuming $\varphi : \partial\Omega \to S^2$ is as above, one of the important problems which is still open is whether smooth harmonic extensions of φ into Ω exist. As a first attempt one may want to minimize the Dirichlet energy in $H^1_\varphi(\Omega, S^2)$ and prove the regularity of the solution. But in fact if we define

$$\mu_\varphi := \inf_{H^1_\varphi(\Omega,S^2)} E(u) \leq \inf_{C^\infty_\varphi(\overline{\Omega},S^2)} E(u) =: \bar{\mu}_\varphi \,,$$

the strict inequality

$$\mu_\varphi < \bar{\mu}_\varphi$$

happens sometimes (See [15]). Thus minimizers of E are not necessarily smooth and we should find other harmonic maps which could be a suitable candidate for a smooth solution. On the other hand R.Schoen and K.Uhlenbeck ([20]) proved that these minimizers are smooth in Ω except on a finite set of points.

In trying to attack this problem, another functional on $H^1_\varphi(\Omega, S^2)$ has been studied which is called the "relaxed energy". In fact, the relaxed energy is the largest sequentially lower semi-continuous functional on $H^1_\varphi(\Omega, S^2)$ which is less than E on $C^\infty_\varphi(\Omega, S^2)$:

Definition 1.1 *The relaxed energy $\mathcal{F}$ of E on $H^1_\varphi(\Omega, S^2)$ is defined to be*

$$\mathcal{F}(u) := \inf\left\{ \liminf_{n\to\infty} E(u_n)\,;\ u_n \in C^\infty_\varphi(\Omega, S^2)\,,\ u_n \rightharpoonup u \right\} . \tag{1.2}$$

When the smooth maps which take φ as their boundary value are weakly sequentially dense in $H^1_\varphi(\Omega, S^2)$ (See [13] and [19]), $\mathcal{F}$ will be well defined. Moreover $\mathcal{F}$ is sequentially lower semi-continuous with respect to the weak topology in $H^1_\varphi(\Omega, S^2)$ and we have

$$\inf_{H^1_\varphi(\Omega,S^2)} \mathcal{F} = \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{1.3}$$

This equation shows the importance of study of $\mathcal{F}$. Since the infimum of $\mathcal{F}$ in $H^1_\varphi(\Omega, S^2)$ is achieved, the question which should be considered then is whether a minimizer of $\mathcal{F}$ is weakly harmonic and to what extent it is regular.

In this direction, based on the results of [7], F.Bethuel, H.Brezis and J.M. Coron (See [5]), established the striking fact that, for $n = 3$, the relaxed energy is given by the following elegant algebraic formula :

$$\mathcal{F}(u) = F(u) := E(u) + 8\pi L(u)$$

where

$$L(u) := \frac{1}{4\pi} \sup_{\substack{\psi : \Omega \to \mathbb{R} \\ |d\psi|_\infty \leq 1}} \left\{ \int_\Omega u^*\omega_V \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega_V \wedge \psi \right\} \tag{1.4}$$

where ω_V is the volume form on S^2 (or can be replaced by any 2-form ω, $\int_{S^2}\omega = 4\pi$). In particular this yields that the critical points of $\mathcal{F}$ are weakly harmonic. However, F.Bethuel and H.Brezis have also shown that the minimizers of

$$\mathcal{F}_\lambda := E + 8\pi\lambda L\,,$$

for $0 \leq \lambda < 1$, are smooth in Ω except on a finite set of points (See[4]).

If $u \in H^1_\varphi(\Omega, S^2)$ is smooth in Ω except on a set of finite points $\{p_1, ..., p_m\}$, with degree d_i at the point p_i, then $L(u)$ is the minimum length of the segments connecting these singularities with respect to the multiplicities (See [7]). In other words

$$L(u) = m_i\left(\sum_{i=1}^m d_i\,[[p_i]],\,\Omega\right)$$

where we define for the integer multiplicity rectifiable 0-current $\mathbf{S}_u = \sum_{i=1}^m d_i\,[[p_i]]$:

$$m_i(\mathbf{S}_u,\Omega) := \inf\left\{\mathbf{M}(\mathbf{T})\,;\, T \in \mathcal{R}_1(\mathbb{R}^3),\ \text{spt}\,\mathbf{T} \subset \overline{\Omega}\,,\ \partial\mathbf{T} = \mathbf{S}_u\right\}.$$

Here we study the same approach for $n > 3$ but this generalization meets new obstacles. One may introduce for ω, any 2-form on S^2 which satisfies $\int_{S^2}\omega = 1$:

$$L(u) := \sup_{\substack{\psi\,\in\,\Lambda^{n-3}(\overline{\Omega})\\ |d\psi|_\infty \leq 1}} \left\{\int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega}\varphi^*\omega\wedge\psi\right\} \tag{1.5}$$

as a generalization of $L(u)$ in the 3-dimensional case. Observe that L is independent of the choice of ω and is continuous on $H^1_\varphi(\Omega, S^2)$ for $\Omega \subset \mathbb{R}^n$ and the functional

$$F(u) := E(u) + 8\pi L(u) \tag{1.6}$$

would still be weakly lower semi-continuous. But we have the following fact :

Theorem I *For every $\Omega \subset \mathbb{R}^4$ and every map $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable into Ω, there exists $u \in H^1_\varphi(\Omega, S^2)$ such that*

$$F(u) < \mathcal{F}(u)\,. \tag{1.7}$$

Moreover there exists a domain $\Omega \subset \mathbb{R}^4$ and $\varphi \in C^\infty(\partial\Omega, S^2)$, smoothly extendable into Ω, for which the following gap phenomenon holds :

$$\inf_{H^1_\varphi(\Omega,S^2)} E < \inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E\,. \tag{1.8}$$

■

The difference with the case $n = 3$ lies in the quantity which $L(u)$ represents. We shall consider a map $u \in H^1_\varphi(\Omega, S^2)$, which is smooth except on a finite union of $(n-3)$-dimensional submanifolds of $\Omega : \{\sigma_1, ...\sigma_m\}$. The degree d_i of u on each σ_i is well defined and we define $\mathbf{S}_u := \sum_{i=1}^m d_i\, [[\sigma_i]]$. Computing $L(u)$, we see that

$$L(u) = \sup_{|d\psi|_\infty \leq 1} \int_{\mathbf{S}_u} \psi \leq \sup_{||d\psi||^*_\infty \leq 1} \int_{\mathbf{S}_u} \psi = m_r(\mathbf{S}_u, \Omega) \tag{1.9}$$

where $||.||^*$ is the co-mass norm on the space of forms and

$$m_r(\mathbf{S}_u, \Omega) := \inf \left\{ \mathbf{M}(\mathbf{T})\,;\, \mathbf{T} \in \mathcal{D}_{n-2}(\mathbb{R}^n)\,, \partial \mathbf{T} = \mathbf{S}_u, \, \text{spt}\, \mathbf{T} \subset \overline{\Omega} \right\}$$

is the mass of the minimal normal (real) current in Ω with boundary $\mathbf{S}_u$. The second equality in (1.9) is due to the fact that there exists always a calibration for minimizing normal currents, which we shall discuss later in this paper (See proposition 2.3). Meanwhile, $m_i(\mathbf{S}_u, \Omega)$, the minimal mass of integer multiplicity rectifiable currents in $\overline{\Omega}$ which are bounded by $\mathbf{S}_u$, is still proportional to the energy needed for removing the singularities of u. Here arises the main question which should be answered if we want to continue as above, that is whether

$$m_r(\mathbf{S}, \Omega) = m_i(\mathbf{S}, \Omega) \quad \forall \mathbf{S} \in \mathcal{R}_{n-3}(\Omega).$$

But in contrast with the case $n = 3$, the answer is no for $n > 3$. In particular, for $n = 4$, there exists a curve $[[\Gamma]]$ in $\mathbb{R}^4$ for which

$$m_r([[\Gamma]]) < m_i([[\Gamma]]).$$

This gap phenomenon was first proved by L.C.Young in [22]. F.Morgan in [17] and B.White in [21] have given other examples of such curves in $\mathbb{R}^4$.

Remark 1.1 *However, in [18], we observed that the critical points of F are still weakly harmonic in $H^1_\varphi(\Omega, S^2)$ and we used this to prove the existence of infinitely many weakly harmonic extensions of φ onto Ω.*

Finally we may search for the amount of energy needed to relax the Dirichlet energy. In section 3 we prove that the topological singular set $\mathbf{S}_u$ of any $u \in H^1_\varphi(\Omega, S^2)$ is the boundary of some integer multiplicity rectifiable current. Then the discussions in this paper suggest that $\mathcal{F}$ coincides with

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega).$$

We can only prove that $\widetilde{F} \leq \mathcal{F}$, the reverse inequality is still an open problem (See proposition 3.1 and the remark following). However, we can prove $\widetilde{F} \geq \mathcal{F}$ when we consider the problem of relaxing the 3-energy of maps into S^3. We will present this example in a forthcoming paper.

2 Preliminaries

2.1 The subspace $R^{\infty}_{\varphi}(\Omega, S^2)$

Definition 2.1 *We say that* $u \in H^1_{\varphi}(\Omega, S^2)$ *is in* $R^{\infty}_{\varphi}(\Omega, S^2)$ *if and only if* u *is smooth except on* $B = \bigcup_{i=1}^m \sigma_i \cup B_0$ *, a compact subset of* Ω*, where* $\mathcal{H}^{n-3}(B_0) = 0$ *and the* σ_i *,* $i = 1, \cdots, m$ *are disjoint smooth embeddings of the open* $(n-3)$*-dimensional unit disk. Moreover we assume that any two* σ_i *and* σ_j *can meet only on their boundaries.*

Remark 2.1 *According to ([2], theorem 2 bis),* $R^{\infty}_{\varphi}(\Omega, S^2)$ *is dense in* $H^1_{\varphi}(\Omega, S^2)$*.*

Definition 2.2 *Let* $u \in H^1_{\varphi}(\Omega, S^2)$*. We define the current* $\mathbf{S}_u \in \mathcal{D}_{n-3}(\Omega)$ *to be the current defined by*

$$\mathbf{S}_u(\alpha) := \int_{\Omega} u^*\omega \wedge d\alpha \qquad \forall \alpha \in \mathcal{D}^{n-3}(\Omega). \tag{2.1}$$

Here $\mathcal{D}^k(\Omega)$ *is the set of smooth k-forms on* Ω *with compact support (See[11], 2.2.3) and* ω *is some 2-form on* S^2 *for which* $\int_{S^2} \omega = 1$*.*

A simple observation shows that the definition of $\mathbf{S}_u$ is independent of the choice of ω due to the fact that the difference of two closed forms on S^2 is exact. The existence of the integral (2.1) is a direct consequence of the following inequality :

$$|u^*\omega| \leq \frac{1}{8\pi}|\nabla u|^2 \qquad \text{a.e. on } \Omega \tag{2.2}$$

where $4\pi\omega = \omega_V$ is the standard volume form of S^2.

Definition 2.3 *Let* $u \in R^{\infty}_{\varphi}(\Omega, S^2)$ *and let* $B = \bigcup \sigma_i \cup B_0$ *be the singular set of* u*. Suppose that each* σ_i *is oriented by a smooth* $(n-3)$*-vectorfield* $\vec{\sigma}_i$*. For* $a \in \sigma_i$ *and* N_a *the 3-dimensional plane orthogonal to* σ_i *at* a*. Consider the 3-disk* $M_{a,\delta} = B_\delta(a) \cap N_a$ *oriented by the 3-vector* $\vec{M}_a$ *such that* $\vec{\sigma}_i(a) \wedge \vec{M}_a = (-1)^n \vec{\xi}_{\mathbb{R}^n}$ *. Then the topological degree of* u *on the 2 dimensional sphere* $\Sigma_{a,\delta} = \partial M_{a,\delta}$ *is well defined and is independent of the choice of* a *for* δ *small enough. We should call this integer the degree of* u *on* σ_i *and denote it by*

$$deg_{\sigma_i} u\,.$$

We shall mention here some useful facts which we have already proved in [18]. Recall that any k-dimensional rectifiable subset $\mathcal{M}$ of $\mathbb{R}^n$ considered with a multiplicity θ and oriented by a unit k-vector field $\vec{\xi}$ defines a rectifiable current as follows

$$\tau(\mathcal{M}, \theta, \vec{\xi})(\alpha) := \int_{\mathcal{M}} < \vec{\xi}, \alpha > \theta \, d\mathcal{H}^k \qquad \forall \alpha \in \mathcal{D}^k(\mathbb{R}^n).$$

Lemma 2.1 *Let* $\omega = \frac{1}{4\pi}\omega_V$ *and* $u \in R^\infty_\varphi(\Omega, S^2)$. *Then the* $(n-2)$*-vectorfield* $\vec{D}(u)$ *defined on* $\Omega \backslash B$ *by the equation*

$$< \vec{D}(u)(x), \Psi > \omega_{\mathbb{R}^n} := u^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda^{n-2}(\mathbb{R}^n) \tag{2.3}$$

is a simple $(n-2)$*-vectorfield tangent to the smooth manifold* $u^{-1}(y)$ *for all regular value* $y = u(x) \in S^2$. *Meanwhile*

$$|\vec{D}(u)| = \frac{1}{4\pi}|J_2 u| \quad a.e. \ on \ \Omega. \tag{2.4}$$

■

An element of $\Lambda_k(\mathbb{R}^n)$ is called *simple* if and only if it equals the exterior product of k vectors of $\mathbb{R}^n$ ([9], 1.6.1).

In the view of lemma 2.1, for any $y \in S^2$ a regular value of $u \in R^\infty_\varphi(\Omega, S^2)$, the current

$$\mathbf{T}^u_y := \tau\left(u^{-1}(y), 1, \frac{\vec{D}(u)}{|\vec{D}(u)|}\right) \tag{2.5}$$

is well defined. Moreover

Proposition 2.1 *Consider* $u \in R^\infty_\varphi(\Omega, S^2)$ *and* $\mathbf{T}^u_y$ *as in* (2.5), *then for almost all* $y \in S^2$, $\mathbf{T}^u_y$ *is a rectifiable current in* $\mathbb{R}^n$ *with support in* $\overline{\Omega}$ *and*

$$\partial \mathbf{T}^u_y = \mathbf{S}_u + \tau\left(\varphi^{-1}(y), 1, \frac{\vec{D}(\varphi)}{|\vec{D}(\varphi)|}\right) \tag{2.6}$$

where the $(n-3)$*-vectorfield* $\vec{D}(\varphi)$ *on* $\partial\Omega$ *is defined by the equation*

$$< \vec{D}(\varphi)(x), \Psi > \omega_{E_x} := \varphi^*\omega(x) \wedge \Psi \quad \forall \Psi \in \Lambda_{n-3}(E_x)$$

where $E_x = T_x(\partial\Omega)$ *is the tangent space to* $\partial\Omega$ *at* x *and* ω_{E_x} *is its unit volume form.*

■

Proposition 2.2 *Let* $u \in R^\infty_\varphi(\Omega, S^2)$ *and* $B = \bigcup_i \sigma_i \cup B_o$ *its singular set. Then*

$$\mathbf{S}_u = \sum_i (deg_{\sigma_i} u)\tau(\sigma_i, 1, \vec{\sigma}_i).$$

■

2.2 Calibrations and minimizing real currents

Let $\mathbf{T}$ be a normal current in $\mathcal{D}_m(\mathbb{R}^n)$ with support in a compact set : K.

Definition 2.4 *The measurable form α in $\Omega^m(\mathbb{R}^n)$ is called to be a calibration for $\mathbf{T}$ in K if*

$$\begin{cases} (i)\ \alpha \text{ is exact}, \\ (ii)\ \|\alpha_{|K}\|_\infty^* \leq 1\,, \\ (iii)\ \mathbf{T}(\alpha) = \mathbf{M}(\mathbf{T})\,. \end{cases} \tag{2.7}$$

We say then that $\mathbf{T}$ is calibrated in K.

We have this interesting proposition which shows the importance of calibrations in the study of minimal currents :

Proposition 2.3 *The real current $\mathbf{T}$ is calibrated in K if and only if it has the minimal mass among all the real currents supported in K and taking the same boundary. Specially for any open bounded set Ω in $\mathbb{R}^n$ and any real flat chain $\mathbf{S}$ in Ω we have*

$$m_r(\mathbf{S}, \Omega) = \sup_{\|d\psi\|_\infty^* \leq 1} \mathbf{S}(\psi). \tag{2.8}$$

■

We omit the proof since it is the same as the proof for ([12], proposition 4.35, p. 59). The interesting fact is that, as a result, a minimal integer multiplicity rectifiable current is calibrated if and only if it is also a minimal real current for the same boundary. The only cases where this always happens are when the minimal current is of dimension or codimension 1 in Ω. In other words if $dim\,\mathbf{S} = 0$ or $n-2$, then

$$m_r(\mathbf{S}, \Omega) = m_i(\mathbf{S}, \Omega). \tag{2.9}$$

For the proof and some counterexamples when the conditions are not satisfied see ([10], section 5). The readers can refer to ([11], vol II, section 1.3.4) for more details. In [1], the authors present an interesting proof of (2.9) for $dim\,\mathbf{S} = n-2$. Also different proofs for the zero dimensional case can be found in [7] and [8]. For other counterexamples see [17], [21] and [22].

2.3 The F-energies

For any 2-form ω on S^2 satisfying $\int_{S^2}\omega = 1$ and $u \in H^1_\varphi(\Omega, S^2)$ we define

$$L(u) := \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{2.10}$$

and

$$L^*(u) := \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ ||d\psi||^*_\infty \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \tag{2.11}$$

where $|.|$ and $||.||^*$ are respectively the euclidean and the co-mass norms on the space of forms. The definitions are independent of the choice of ω (See [18]), so from now on we put $\omega = (1/4\pi)\omega_V$.

Remark 2.2 *L and L^* are both continuous with respect to the H^1 norm in $H^1_\varphi(\Omega, S^2)$. The proof is the same as for the case $n = 3$ in [5].*

We have

Lemma 2.2 *For any $u \in H^1_\varphi(\Omega, S^2)$, $\mathbf{S}_u$ is a real flat chain. Moreover we have*

$$L(u) \leq L^*(u) = m_r(\mathbf{S}_u, \Omega)\,. \tag{2.12}$$

■

Proof : Set

$$\mathbf{D}_u(\alpha) := \int_\Omega u^*\omega \wedge \alpha \qquad \forall \alpha \in \mathcal{D}^{n-2}(\Omega).$$

Since by (2.2) we have $8\pi\mathbf{M}(\mathbf{D}_u) \leq E(u)$, $\mathbf{D}_u$ is a normal current. Moreover, by definition, $\mathbf{S}_u = \partial\mathbf{D}_u$, so $\mathbf{S}_u$ is a real flat chain. We have, using (2.8),

$$\begin{aligned} m_r(\mathbf{S}_u - \mathbf{S}_v, \Omega) &= \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ ||d\psi||^*_\infty \leq 1}} (\mathbf{S}_u - \mathbf{S}_v)(\psi) \\ &\leq C||\nabla u||_2||\nabla v||_2(||\nabla u - \nabla v||_2), \end{aligned}$$

where the last inequality is obtained by the same method as in ([5], theorem 1). As a result $m_r(\mathbf{S}_u, \Omega)$ is continuous with respect to the strong topology in $H^1_\varphi(\Omega, S^2)$.

On the other hand, if $u \in R^\infty_\varphi(\Omega, S^2)$, using the co-area formula and proposition 2.1 successively we obtain

$$\begin{aligned} \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi &= \int_\Omega u^*\omega \wedge d\psi - \int_\Omega \phi^*\omega \wedge d\psi \\ &= \int_{S^2} (\mathbf{T}^u_w(d\psi) - \mathbf{T}^\phi_w(d\psi))\, dw \\ &= \mathbf{S}_u(\psi) \end{aligned}$$

where ϕ is any smooth extension of φ into Ω. This implies

$$\begin{aligned} L^*(u) &= \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ \|d\psi\|_\infty^* \leq 1}} \left\{ \int_\Omega u^*\omega \wedge d\psi - \int_{\partial\Omega} \varphi^*\omega \wedge \psi \right\} \\ &= \sup_{\|d\psi\|_\infty^* \leq 1} \mathbf{S}_u(\psi) = m_r(\mathbf{S}_u, \Omega). \end{aligned}$$

Since $L^*(u)$ and $m_r(\mathbf{S}_u, \Omega)$ are continuous in H^1-norm and considering the fact that $R^\infty_\varphi(\Omega, S^2)$ is dense in $H^1_\varphi(\Omega, S^2)$ for the strong topology, we deduce the equality in (2.12). Moreover, $L \leq L^*$ as $\|\psi\|^*_\infty \leq \|\psi\|_\infty$ for all differential forms. ∎

Definition 2.5 *We define the F-energies to be*

$$F(u) := E(u) + 8\pi L(u)$$

and

$$F^*(u) := E(u) + 8\pi L^*(u).$$

2.4 Sequentially weak density of smooth maps in $H^1_\varphi(\Omega, S^2)$

Let us recall some facts about maps in $R^\infty_\varphi(\Omega, S^2)$:

Proposition 2.4 *There exists $C > 0$ such that for all $u \in R^\infty_\varphi(\Omega, S^2)$ we have*

$$8\pi m_i(\mathbf{S}_u) \leq E(u) + C. \tag{2.13}$$

Moreover there exists a sequence $u_m \in R^\infty_\varphi(\Omega, S^2)$ such that

$$\begin{cases} \mathbf{S}_{u_m} = 0 \\ u_m = u \text{ on } K_m \\ \mu(K_m) \to 0 \text{ as } m \to \infty \\ E(u_m) \leq E(u) + 8\pi m_i(\mathbf{S}_u) + \dfrac{1}{m} \\ u_m \rightharpoonup u \text{ in } H^1 \end{cases} \tag{2.14}$$

∎

(2.13) is proved in [1]. In ([2], section VI), the author, suggesting (2.14) and considering (2.13), remarked that smooth maps are sequentially dense in $H^1_\varphi(\Omega, S^2)$ for the weak topology, as in the case $n = 3$ (See [3]). Recent developments by F.Hang and

F.H.Lin have shown that this argument should be modified for when the domain is not contractible. They remarked that "$\mathbf{S}_u = 0$" is not always the sufficient condition for the strong approximability of $u \in H^1(M, S^2)$ by smooth maps and we should consider global topological obstructions too (See [14]). But the arguments used in [6] work locally and therefore if $\mathbf{B}^n$ is the n-dimensional unit disk in $\mathbb{R}^n$, for any map $u \in H^1_\varphi(\mathbf{B}^n, S^2)$, there exists a sequence of smooth maps $u_m \in C^\infty_\varphi(\mathbf{B}^n, S^2)$ such that

$$\begin{cases} (i)\, u_m \rightharpoonup u \text{ in } H^1 \\ \\ (ii)\, E(u_m) \le 2E(u) + C + O(\frac{1}{m}) \end{cases} \tag{2.15}$$

We will present our method for proving (2.14) in a forthcoming paper where we will treat the question of sequentially weak density of smooth maps in Sobolev spaces between manifolds ([19]).

3 A lower bound for the relaxed energy

Proposition 3.1 *Let $u \in H^1_\varphi(\Omega, S^2)$, then $\mathbf{S}_u$ is the boundary of some integer multiplicity rectifiable current. Set*

$$\widetilde{F}(u) := E(u) + 8\pi m_i(\mathbf{S}_u, \Omega). \tag{3.1}$$

$\widetilde{F}$ is lower semi-continuous with respect to the weak topology on $H^1_\varphi(\Omega, S^2)$ and

$$\widetilde{F}(u) \le \mathcal{F}(u)\,, \quad \forall u \in H^1_\varphi(\Omega, S^2). \tag{3.2}$$

■

Remark 3.1 *We do not prove that $\widetilde{F}$ is the relaxed energy. A stronger result for the case $\Omega = \mathbf{B}^n$ would be to show that $m_i(\mathbf{S}_u, \mathbf{B}^n)$ is continuous on $H^1_\varphi(\mathbf{B}^n, S^2)$, which is still an open problem (Compare with Remark 2.2 and lemma 2.2).*

Proof : For the sake of simplicity we prove the proposition for $\Omega = \mathbf{B}^n$, the n-dimensional unit disk. For the general case we can replace smooth maps by maps satisfying the condition $\mathbf{S}_{u_m} = 0$.

Let $u \in H^1_\varphi(\Omega, S^2)$ and consider a sequence of smooth maps converging weakly to u as in (2.15). Since u_m is smooth, $\partial G_{u_m} = 0$, where G_{u_m} is the graph of u_m. Also since the Dirichlet energy is regular (See [11], vol II, section 5.2.1), the G_{u_m} are equibounded in mass. By the Compactness theorem, there is an integer multiplicity rectifiable n-current T supported in $\Omega \times S^2$ such that $G_{u_m} \rightharpoonup T$ up to some subsequence. By ([11], vol I, section 5.5.2, proposition 3), $G_{u_m} \in cart^{2,1}(\Omega \times S^2)$ for all m. So by the closure theorem

([11], vol I, section 5.5.2, theorem 6) and the Structure theorem ([11], vol II, section 5.2.1) we have

$$T = G_u + L_T \times [[S^2]] \in cart^{2,1}(\Omega \times S^2)$$

while L_T is an $(n-2)$-dimensional integer multiplicity rectifiable current in Ω. From ([11], vol II, section 1.2.4, proposition 15) and (2.15) we deduce that

$$8\pi \mathbf{M}(L_T) \leq E(u) + C. \tag{3.3}$$

Now let π and $\hat{\pi}$ be the respective projections of $\Omega \times S^2$ on Ω and S^2. Since $\partial T = 0$, for any 2-form ω on S^2 and any compactly supported $(n-3)$-form α in Ω we have

$$\int_\Omega u^*\omega \wedge d\alpha = G_u(\pi^*(d\alpha) \wedge \hat{\pi}^*\omega) = \partial G_u(\pi^*\alpha \wedge \hat{\pi}^*\omega) = -\partial L_T(\alpha).$$

So $\mathbf{S}_u = \partial(-L_T)$, which proves the first claim of the proposition. Moreover, as a consequence, $\widetilde{F}$ is well defined for the maps in $H^1_\varphi(\Omega, S^2)$.

Let u_m be a sequence of maps in $H^1_\varphi(\Omega, S^2)$ converging weakly to u. We will prove that

$$\widetilde{F}(u) \leq \liminf_{m\to\infty} \widetilde{F}(u_m). \tag{3.4}$$

Put

$$\beta := \liminf_{m\to\infty} \widetilde{F}(u_m).$$

Passing to some subsequence of u_m if necessary, we have $\widetilde{F}(u_m) \to \beta < +\infty$. Let $-L_m$ be the mass minimizing integer multiplicity rectifiable current taking $\mathbf{S}_{u_m}$ as its boundary. The u_m are equibounded in energy while the L_m are equibounded in mass. So, using the same arguments as above, we see that the cartesian currents

$$T_m := G_{u_m} + L_m \times [[S^2]]$$

converge to some current $T := G_u + L \times [[S^2]]$ in $cart^{2,1}(\Omega \times S^2)$, up to a subsequence. By ([11], vol II, section 1.2.4, proposition 15) we get

$$\begin{aligned}
\widetilde{F}(u) &= E(u) + 8\pi m_i(\mathbf{S}_u) \leq E(u) + 8\pi \mathbf{M}(L) \\
&\leq \liminf_{m\to\infty}(E(u_m) + 8\pi \mathbf{M}(L_m)) \\
&= \liminf_{m\to\infty}(E(u_m) + 8\pi m_i(\mathbf{S}_{u_m})) \\
&= \beta.
\end{aligned}$$

This proves (3.4). Thus $\widetilde{F}$ is lower semi-continuous with respect to the weak topology. (3.2) follows immediately as $\widetilde{F}$ coincides with E on smooth maps. ■

4 Proof of theorem 1

4.1 Proof of (1.7)

Regarding lemma 2.2 and proposition 3.1, it suffices to prove the existence of a map $u \in R^\infty_\varphi(\Omega, S^2)$ which satisfies

$$m_r(\mathbf{S}_u, \Omega) < m_i(\mathbf{S}_u, \Omega). \tag{4.1}$$

This happens for $n = 4$. Specially there is a curve Γ in $\mathbb{R}^4$ for which $m_r([[\Gamma]]) < m_i([[\Gamma]])$ (See [22], [10] and [11], vol II for more details). For any $\Omega \subset \mathbb{R}^4$ and boundary data φ, we can construct a map $u \in R^\infty_\varphi(\Omega, S^2)$ which is smooth except on such a curve, supported in a small ball in Ω. The method is almost the same as the one used by the authors in [1] for constructing a map with prescribed singularities and constant boundary value, so we will not expose the details in this paper. This map will satisfy (4.1).

4.2 Sketch of the proof for (1.8)

a) For $0 < \delta' < \delta$, we construct a domain $\Omega_{\delta,\delta'} \subset \mathbb{R}^4$ and a map $\varphi_{\delta,\delta'} \in C^\infty(\partial\Omega_{\delta,\delta'}, S^2)$ which is extendable onto $\Omega_{\delta,\delta'}$. We put

$$H^1_{\delta,\delta'} := H^1_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$$

and

$$C^\infty_{\delta,\delta'} := H^1_{\delta,\delta'} \cap C^\infty(\Omega_{\delta,\delta'}, S^2).$$

b) We prove that

$$\inf_{H^1_{\delta,\delta'}} F^* - \inf_{C^\infty_{\delta,\delta'}} E = O(\delta) - k$$

when $k > 0$.

c) Regarding the fact that $F \leq F^*$ the theorem is proved by choosing δ small enough.

4.3 Construction of $\Omega_{\delta,\delta'}$

Let $\mathbf{B}$ be an integer multiplicity m-rectifiable current in $\mathbb{R}^n$, without boundary. Put

$$m_i(\mathbf{B}) := \min\{\mathbf{M}(\mathbf{T})\,;\, \partial\mathbf{T} = \mathbf{B},\ \mathbf{T} \in \mathcal{R}_{m+1}(\mathbb{R}^n)\} \tag{4.2}$$

where $\mathbf{M}(\mathbf{T})$ is the mass of $\mathbf{T}$. By [22] there exists Γ, a closed curve on $K \subset \mathbb{R}^4$, a surface homeomorph to the Klein bottle and $\mathbf{A}$, integer multiplicity rectifiable surface in $\mathbb{R}^4$ such that :

$$\begin{cases} (i)\ \partial\mathbf{A} = 2[[\Gamma]] \\ (ii)\ \mathbf{M}(\mathbf{A}) < 2m_i([[\Gamma]]) \\ (iii)\ \mathrm{spt}\mathbf{A} = K \end{cases} \tag{4.3}$$

where

$$[[\Gamma]] := \tau(\Gamma, 1, \vec{v})$$

is the integer multiplicity rectifiable current based on Γ and oriented by the unit tangent vectorfield $\vec{v}$. By slight modifications of Γ and K around their singular subsets, we may consider them to be smooth. Let $\vec{n}$ be a smooth normal vectorfield on $K \subset \mathbb{R}^4$. We recall that $\Gamma \subset K$ and we put :

$$\Sigma_\delta := \{x + t\vec{n}(x)\,;\, 0 \leq t \leq \delta\,,\, x \in \Gamma\}$$

and

$$\Gamma_\delta := \{x + \delta\vec{n}(x)\,;\, x \in \Gamma\}.$$

We observe that for Σ_δ and Γ_δ suitably oriented and δ sufficiently small we have :

$$\partial[[\Sigma_\delta]] = [[\Gamma_\delta]] - [[\Gamma]]. \tag{4.4}$$

Let V_δ be the tubular neighborhood of Γ_δ :

$$V_\delta := \{y \in \mathbb{R}^4\,;\, d(y, \Gamma) \leq \delta\}.$$

For each $y \in \Gamma$ and $0 < \delta' < \delta$ let $B(\delta, \delta', y)$ be the 2-dimensional disk in $\mathbb{R}^4$ centered at y and with radius δ' which is orthogonal to Σ_δ and observe that

$$B(\delta, \delta') := \bigcup_{y \in \Gamma} B(\delta, \delta', y)$$

is a 3-dimensional submanifold of $\mathbb{R}^4$. We shall construct $\Omega_{\delta,\delta'}$ such that $B(\delta, \delta') \subset \Omega_{\delta,\delta'}$.

Let T be a smooth surface such that

$$\begin{cases} (i)\, \partial T = [[\Gamma_\delta]] \\ (ii)\, T \cap B(\delta, \delta') = \emptyset \\ (iii)\, \vec{n}(x) \text{ is the outward tangent to } T \text{ at } x + \delta\vec{n}(x) \in \partial T\,,\ \forall x \in k. \end{cases} \tag{4.5}$$

Such T exists : As $\pi_1(\mathbb{R}^4 \backslash V_\delta) = 0$, there exists some smooth $T_0 \subset \mathbb{R}^4 \backslash V_\delta$ such that $\partial T_0 = [[\Gamma]]$. So if $T_1 = \Sigma_\delta \cup T_0$ we get $\partial T_1 = [[\Gamma_\delta]]$. T is obtained by smoothing T_1 in a neighborhood of Γ. Let $\vec{e_1}, \vec{e_2}$ be 2 smooth orthonormal vectorfields on T such that for each $y \in \Gamma_\delta$, $\vec{e_1}(y)$ and $\vec{e_2}(y)$ are tangent to $B(\delta, \delta', y)$. We put

$$U_\delta := \{x + t_1\vec{e_1}(x) + t_2\vec{e_2}(x)\,;\ (t_1^2 + t_2^2)^{\frac{1}{2}} \leq \delta\}.$$

We choose δ small enough and some $\delta' < \delta$ such that

$$B(\delta, \delta') \cap W_{\delta'} = \emptyset$$

where $W_{\delta'} := \{x \in \mathbb{R}^4 \,;\ d(x,K) \leq \delta'\}$. This is possible since Γ_δ has no intersection with K.

For every $x \in \Gamma$, $y = x + \delta\vec{n}(x)$, let $C(\delta,\delta',y)$ be the cone with the vertex x and the base $B(\delta,\delta',y)$ and put

$$C(\delta,\delta') := \bigcup_{y \in \Gamma_\delta} C(\delta,\delta',y).$$

We define the map $\pi : C(\delta,\delta') \to B(\delta,\delta')$ as follows : For every $z \in C(\delta,\delta',y)$, $\pi(z)$ is the intersection of the line $x-z$ and the disk $B(\delta,\delta',y)$, where x is the vortex of the cone $C(\delta,\delta',y)$. Then we put

$$\Omega_{\delta,\delta'} := C(\delta,\delta') \cup U_{\delta'} \cup W_{\delta'}. \tag{4.6}$$

$\Omega_{\delta,\delta'}$ is a domain in $\mathbb{R}^4$ which contains tubular neighborhoods of K and T while $\partial\Omega_{\delta,\delta'}$ contains the set $B(\delta,\delta')$.

4.4 Construction of $\varphi_{\delta,\delta'}$

Let B be the unit disk in $\mathbb{R}^2$ and $\nu : B \to S^2$ be the smooth covering map as defined in [4], which satisfies these conditions :

$$\begin{cases} (i)\ \nu|_{\partial B} = const. = e \in S^2\,,\ \nu(0) = -e \\ (ii)\ \displaystyle\int_B |\nabla\nu|^2 = 4\pi \\ (iii)\ \text{For } z \neq e \text{ in } S^2,\ \#\omega^{-1}(z) = 1 \text{ and } \deg(\nu,B,0) = 1. \end{cases} \tag{4.7}$$

We define the map $\phi_{\delta,\delta'} \in C^\infty(\Omega_{\delta,\delta'},S^2)$ as follows :

$$\phi_{\delta,\delta'}(z) := \begin{cases} \nu\left(\left(\dfrac{t_1}{\delta'},\dfrac{t_2}{\delta'}\right)\right) & \text{if } z = x + t_1\vec{e_1} + t_2\vec{e_2} \in U_{\delta'} \\ e & \text{if } z \notin U_{\delta'} \end{cases}$$

And we put

$$\varphi_{\delta,\delta'} := \phi_{\delta,\delta'}|_{\partial\Omega_{\delta,\delta'}}\,.$$

4.5 Estimation for $\inf E$ on $C^\infty_{\delta,\delta'}$

Let $u \in C^\infty_{\delta,\delta'}$. By (2.2), (2.4) and the co-area formula we get :

$$\begin{aligned} \int_{\Omega_{\delta,\delta'}} |\nabla u|^2 &\geq 8\pi \int_{\Omega_{\delta,\delta'}} |u^*\omega| = 2\int_{\Omega_{\delta,\delta'}} |J_2 u| \\ &= 2\int_{S^2} dw \int_{u^{-1}(w)} 1 = 2\int_{S^2} \mathbf{M}(\mathbf{T}^u_w)\,dw. \end{aligned} \tag{4.8}$$

On the other hand, considering the propositions 2.1 and 2.2, we have

$$\partial \mathbf{T}^u_w = [[\varphi^{-1}_{\delta,\delta'}(w)]]. \tag{4.9}$$

However, for each $w \neq e \in S^2$, there exists some surface $S_{w,\delta} \subset B(\delta, \delta')$ such that

$$\partial[[S_{w,\delta}]] = [[\varphi^{-1}_{\delta,\delta'}(w)]] - [[\varphi^{-1}_{\delta,\delta'}(-e)]] = [[\varphi^{-1}_{\delta,\delta'}(w)]] - [[\Gamma_\delta]]$$

for suitable orientations. Using this and regarding (4.4) we get

$$\begin{aligned} |\, m_i\left([[\Gamma]]\right) - m_i\left([[\varphi^{-1}_{\delta,\delta'}(w)]]\right) | \ &\leq |\, m_i\left([[\Gamma]] - [[\varphi^{-1}_{\delta,\delta'}(w)]]\right) | \\ &\leq |\Sigma_\delta| + |B(\delta, \delta')| = O(\delta). \end{aligned}$$

This estimation, combined with (4.8) and (4.9) gives :

$$E(u) \geq 2 \int_{S^2} m_i\left([[\varphi^{-1}_{\delta,\delta'}(w)]]\right) dw = 8\pi m_i([[\Gamma]]) + O(\delta) \quad \forall u \in C^\infty_{\delta,\delta'}$$

and as a result

$$\inf_{C^\infty_{\delta,\delta'}} E \geq 8\pi m_i\left([[\Gamma]]\right) + O(\delta)\,. \tag{4.10}$$

4.6 Estimation for $\inf F^*$ on $H^1_{\delta,\delta'}$

We put for $z \in \Omega_{\delta,\delta'}$:

$$u_{\delta,\delta'} := \begin{cases} \varphi_{\delta,\delta'}(\pi(z)) & \text{if } z \in C(\delta, \delta') \\ e & \text{if } z \notin C(\delta, \delta') \end{cases}$$

We have for $K > 0$ independent of δ and δ' :

$$\begin{cases} |\nabla u_{\delta,\delta'}| = 0 & \text{on } \Omega_{\delta,\delta'} \backslash C(\delta, \delta') \\ |\nabla u_{\delta,\delta'}| \leq |\nabla \varphi_{\delta,\delta'}|\, |\nabla \pi| \leq \dfrac{K}{\delta'} & \text{on } C(\delta, \delta') \\ u_{\delta,\delta'}|_{\partial\Omega_{\delta,\delta'}} = \varphi_{\delta,\delta'} \end{cases} \tag{4.11}$$

Therefore

$$E(u_{\delta,\delta'}) \leq \int_{C(\delta,\delta')} \frac{K^2}{\delta'^2} \leq \frac{K^2}{\delta'^2} |C(\delta, \delta')| = O(\delta). \tag{4.12}$$

As a result $u_{\delta,\delta'} \in H^1_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$. We should estimate $L^*(u_{\delta,\delta'})$: Pay attention that $u_{\delta,\delta'} \in R^\infty_{\varphi_{\delta,\delta'}}(\Omega_{\delta,\delta'}, S^2)$ as it is smooth on $\Omega_{\delta,\delta'} \backslash \Gamma$. Proposition 2.2 and a simple topological observation show that if Γ is suitably oriented we have

$$\mathbf{S}_{u_{\delta,\delta'}} = [[\Gamma]].$$

Recall that $\partial \mathbf{A} = 2[[\Gamma]]$ and that $\text{spt}\mathbf{A} \subset W_{\delta'} \subset \Omega_{\delta,\delta'}$ (See (4.6)) . So referring to lemma 2.2 and using (4.12) we have :

$$F^*(u_{\delta,\delta'}) = E(u_{\delta,\delta'}) + 8\pi m_r(\mathbf{S}_{u_{\delta,\delta'}}, \Omega_{\delta,\delta'}) \leq 4\pi \mathbf{M}(\mathbf{A}) + O(\delta)$$

and as a result

$$\inf_{H^1_{\delta,\delta'}} F^* \leq F^*(u_{\delta,\delta'}) \leq 4\pi \mathbf{M}(\mathbf{A}) + O(\delta)\,. \tag{4.13}$$

4.7 End of the proof

Combining (4.10) and (4.13) we get

$$\inf_{H^1_{\delta,\delta'}} F^* - \inf_{C^\infty_{\delta,\delta'}} E = O(\delta) - 4\pi(2m_i\left([[\Gamma]]\right) - \mathbf{M}(\mathbf{A}))\,.$$

But regarding (4.3) we know that

$$2m_i\left([[\Gamma]]\right) - \mathbf{M}(\mathbf{A}) > 0\,.$$

Therefore by choosing δ small enough, for $\Omega = \Omega_{\delta,\delta'}$ and $\varphi = \varphi_{\delta,\delta'}$ we get :

$$\inf_{H^1_\varphi(\Omega,S^2)} F^* - \inf_{C^\infty_\varphi(\Omega,S^2)} E < 0\,.$$

This shows that

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{C^\infty_\varphi(\Omega,S^2)} E \tag{4.14}$$

as $F \leq F^*$.

We also claim that for suitable δ,

$$\inf_{H^1_\varphi(\Omega,S^2)} E < \inf_{H^1_\varphi(\Omega,S^2)} F.$$

Since F is coercive and weakly lower semi-continuous (As we mentioned in [18], the proof is as in [5] for $n = 3$), its minimum is achieved by some $v \in H^1_\varphi(\Omega, S^2)$. If $\mathbf{S}_v \neq 0$ we have

$$L(v) = \sup_{\substack{\psi \in \Lambda^{n-3}(\overline{\Omega}) \\ |d\psi|_\infty \leq 1}} \int_\Omega \mathbf{S}_v > 0\,,$$

which implies :

$$\inf_{H^1_\varphi(\Omega,S^2)} E \leq E(v) < E(v) + 8\pi L(v) = F(v) = \inf_{H^1_\varphi(\Omega,S^2)} F$$

and the claim is proved. Otherwise, $\mathbf{S}_u = 0$. So $F(v) = E(v)$. If our claim is not true, v would minimize E in $H^1_\varphi(\Omega, S^2)$. By partial regularity theory of [20], $v \in R^\infty_\varphi(\Omega, S^2)$. As a result

$$\inf_{H^1_\varphi(\Omega,S^2)} F = F(v) = E(v) \geq \inf_{S^0_\varphi(\Omega,S^2)} E\,,$$

where

$$S^0_\varphi(\Omega, S^2) := \{u \in R^\infty_\varphi(\Omega, S^2); \mathbf{S}_u = 0\}.$$

On the other hand, using the same arguments as above we can prove that

$$\inf_{H^1_\varphi(\Omega,S^2)} F < \inf_{S^0_\varphi(\Omega,S^2)} E$$

for suitable $\delta > 0$. This leads to a contradiction. So our claim is true and this completes the proof of theorem I. ∎

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